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Distributed Dissipative Model Predictive Control for Process Networks with Imperfect Communication

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Results are developed to ensure stability of a dissipative distributed model predictive controller in the case of structured or arbitrary failure of the controller communication network; bounded errors in the communication may similarly be handled. Stability and minimum performance of the process network is ensured by placing a dissipative trajectory constraint on each controller. This allows for the interaction effects between units to be captured in the dissipativity properties of each process, and thus, accounted for by choosing suitable dissipativity constraints for each controller. This approach is enabled by the use of quadratic difference forms as supply rates, which capture detailed dynamic system information. A case study is presented to illustrate the results. © 2014 American Institute of Chemical Engineers *AIChE J*, 60: 1682–1699, 2014
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Introduction

Model predictive control (MPC) is one of the few advanced control techniques which has achieved wide spread use in the chemical industry,[1,2] due to its ability to handle constraints (both soft and hard), multivariable problems, and generate an optimal control action. Constraints, along with large scale and strong interactions (potentially inducing time-scale separation) are key problems in the control of chemical process networks.[3,4] These interactions between process units are due to material recycle and heat integration, which are common in modern chemical plants as a result of designs based upon steady-state efficiency concerns. These recycle loops can be thought of as positive feedback interconnections from a control point of view, which are well known present challenges in control practice.

As such, a scalable approach to MPC for process networks which takes interaction effects into account is desired. Although centralized approaches can handle interaction effects, they suffer from poor scalability due to the computational complexity involved in modeling and design. Conversely, decentralized MPC offers a more scalable approach, although it may lead to poor performance or even instability if interaction effects are not accounted for.[5] Due to these short comings, distributed MPC has received much attention in the literature, as it attempts to realize the benefits of centralized and decentralized approaches, some recent examples include[6,7] and those by Rawlings and coworkers,[8,9] and Christofides and coworkers,[10–12] a recent review is also available.[13]

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In our previous work,[14] a dissipativity-based approach to noncooperative distributed MPC for systems with constant interconnection topology was presented. A dissipativity ensuring constraint is imposed on the individual controllers such that the closed loop process network satisfies a desired dissipativity condition. Dynamic supply rates in the form of quadratic difference forms (QdFs) were used to ensure stability and minimum performance bounds on the process network.

In this work, the controller network is allowed to change to model the effect of failure of the controller communication network and network noise (i.e., due to quantization error). A "plant-wide connective stability" (PCS) condition has been recently presented,[15] which ensures stability of the plant-wide closed loop system for arbitrary failures in the controller network. The framework presented in the current article ensures plant-wide stability for arbitrary failures in the controller network, and/or structured failures, thus, it leads to less conservative results as compared to Ref. 15. This has application to the case where some communication links are more prone to failure (for example, wireless links) than others, thus, stability may be assured for the less robust communication links only. This has applications when some process units are geographically separated, as may be the case in minerals processing applications or renewable energy generation. Changes in the process network may similarly be handled within this framework. It should be noted that communication disruptions have been studied in the context of distributed MPC,[16] whereby a two level control architecture is implemented along with a feasibility check to determine if the communication is reliable.

Dissipative systems theory is a natural tool for the analysis of large-scale systems as dissipativity is fundamentally an input–output property of systems, thus allowing interaction effects to be captured. Furthermore, the dissipativity of the

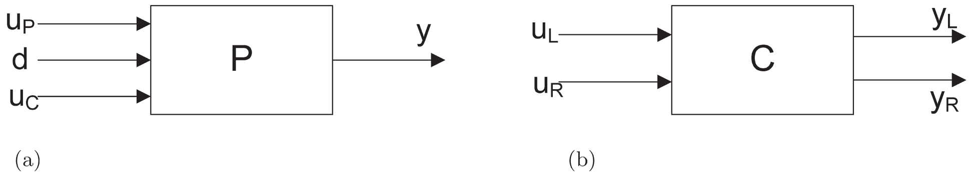


Figure 1. Partitioning of subsystem variables.

(a) Partitioning of process manifest variables. (b) Partitioning of controller variables.

plant-wide system can be determined as a linear combination of that of the individual subsystems regardless of the interconnection topology. Some approaches to the analysis and control of large-scale systems include Refs. 17–20. Parallel to this, there have been some dissipativity-based approaches to MPC for single systems in the recent literature. Robust MPC for systems with dissipative uncertainty has also been studied.[21,22] Whilst an approach to MPC for nonlinear systems has been presented based on the passivity theorem to ensure closed-loop stability.[23] Chen and Scherer placed a dissipativity ensuring constraint on the online MPC algorithm, which guarantees minimum $\mathcal{H}_\infty$ performance.[24,25]

A key point of difference of our approach from those above is that it is based upon the dissipativity of the open-loop processes and controllers, not the closed-loop subsystems. This also allows for the changes in the controller communication network to be treated explicitly, so that the controller network can be designed to deal with problems of certain communication paths that are more prone to errors. In contrast with existing approaches, for example Ref. 15, where the plant-wide stability conditions under communication errors based on the closed-loop plant-wide models and as such the effects of the controller communication network are hidden by the analysis. The approach presented in the current article is facilitated by the use of QdFs as supply rates, which provide sharper stability conditions and, therefore, less conservative results.[26] Such an approach may be prohibitively conservative using the conventional (Q, S, R) supply rates. It should also be noted that the above dissipative MPC approaches are for the control of single systems, not for distributed control as in this article. Additionally, the types of dissipativity used in these works may be seen as special cases of that which can be described by QdFs in the proposed approach.

Some notation used in the remainder of this article is briefly introduced. $A > (\geq) 0$ for symmetric matrices A, means that A is positive definite (semidefinite), similarly $A > B \Longleftrightarrow (A-B) > 0$, the negative definite and semidefinite cases are analogously defined. $\mathrm{diag}(A_1, \ldots, A_n)$ denotes the formation of a block matrix with A_i as its ith diagonal entry. The maximum singular value of a matrix A is denoted by $\bar{\sigma}(A)$. The Euclidean norm of a vector is denoted by $||.||_2$. σ denotes the forward shift operator, that is, $\sigma^k x(t)=x(t+k)$. $\phi(\zeta,\eta) \in \mathbb{R}^{n\times m}(\zeta,\eta)$ denotes an $n\times m$ dimensional two variable polynomial matrix in the indeterminates ζ and η with real coefficients. The degree of such a matrix, denoted by $\deg(\phi)$, is defined as the maximum power of ζ and η appearing in any element of $\phi(\zeta,\eta)$.

The remainder of this article is structured as follows: in the next section the problem definition is stated; following this, the relevant concepts of dissipativity and QdFs are revised. Then, the dissipativity properties of the process network are formulated based on that of the individual subsystems and the interconnection topology followed by dissipativity-based stability and performance conditions. Next, the MPC algorithm is presented along with the offline optimization problem to determine the required dissipativity constraints for the individual controllers. The article is concluded by an illustrative example and discussion.

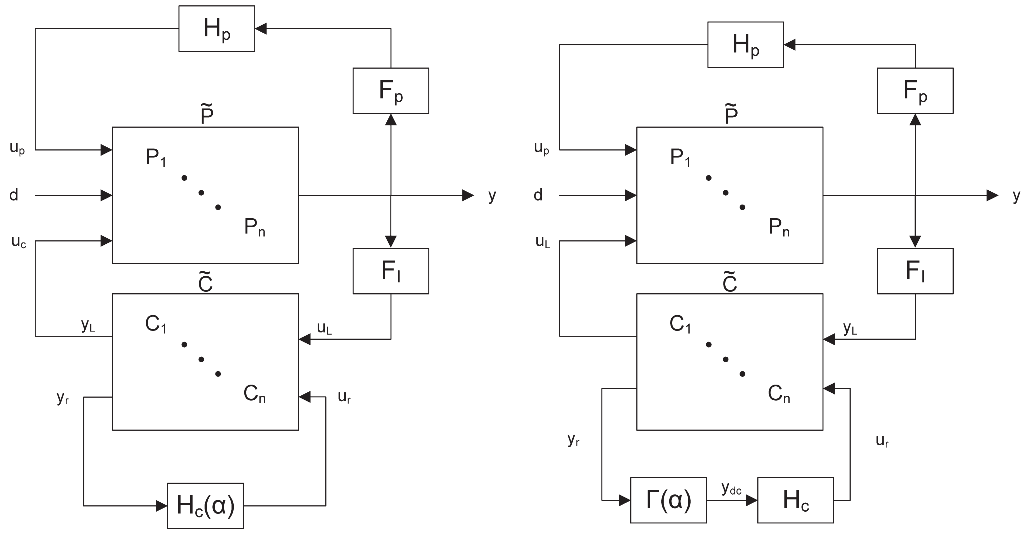
Problem Definition

Consider a chemical process network of n linear time-invariant (LTI) process units connected by a process network with constant topology. In this article, we are concerned with a regulating problem around a given set point (i.e., to produce a desired product). As such, linear (or linearized) models of individual systems are used, with the ith process represented as in (1). The justification for this is that in the operating region of interest, the actual subsystems will behave similarly to their linear approximations. Partitioning the input to each process unit into the input from interconnected processes u_p, external disturbances d, and manipulated input u_c, the ith process unit P_i has minimal state representation

$$P_i : \left\{ \begin{pmatrix} x_i(k+1) \\ y_i(k) \end{pmatrix} = \begin{pmatrix} A_i & B_{1_i} & B_{2_i} & B_{3_i} \\ C_i & D_{1_i} & D_{2_i} & D_{3_i} \end{pmatrix} \begin{pmatrix} x_i(k) \\ u_{c_i}(k) \\ u_{p_i}(k) \\ d_i(k) \end{pmatrix} \right. \quad (1)$$

This partitioning of the individual process units inputs and outputs is shown in Figure 1a. The control problem is to design n distributed model predictive controllers to yield a stable closed-loop plant-wide system with guaranteed minimum $\mathcal{H}_\infty$ performance bounds with given hard input and soft output constraints. These distributed controllers may be viewed as having local and remote ports which interface with their local process units and other controllers, respectively. This partitioning of the controller inputs and outputs is shown in Figure 1b. This distributed control system must achieve these specifications in the case that the communication between certain controllers fails, that is, no signals are sent from the ith controller to the jth controller for an arbitrary length of time (including a possibly infinite length of time) before (possibly) resuming. In addition, these specifications must also be achieved in the case of errors in controller communication that may be modeled as bounded additive noise. That is, if $v_{ij}(k)$ is the signal sent from the ith controller to the jth controller at time k, the jth controller receives $v_{ij}(k)+\delta(k)$, where $\delta(k) \in [\underline{\delta}, \bar{\delta}], \forall k \in \mathbb{R}^+$. An example of such noise is quantization error due to the word length of the communication. It is assumed that the designer knows which communication links are prone to failure and/or noise; which may either be a subset, or all, of the controller communication links.

The overall process and controller networks are depicted in Figure 2a, the systems $\tilde{P}$ and $\tilde{C}$ represent each process and controller stacked diagonally, respectively. That is,



(a) (b)

Figure 2. Network view of large-scale systems with distributed controller.

(a) Large-scale system with constant controller network. (b) Large-scale system with varying controller network.

$\tilde{P}=\text{diag}(P_1,\ldots,P_n)$. The matrices F_p and F_I represent "filters" which select the interconnecting and measured outputs, respectively, these are considered in this work as constant matrices with elements either 0 or 1. The case where the controller communication network changes during operation are handled by changing the $H_c(\alpha)$ matrix as described below. The matrix H_p represents the process network topology; a novel feature of this work is that $H_c(\alpha)$ is allowed to be a linear function of n scalar parameters $\alpha_i, i\in[1,n]$ as follows

$$H_c(\alpha)=H_{c_0}+H_{c_1}\alpha_1+\cdots+H_{c_n}\alpha_n \tag{2}$$

$$=\tilde{H}_c\Gamma(\alpha) \tag{3}$$

where $\alpha_i\in\mathcal{A}\forall i$, a convex set, and the H_{c_i} are constant matrices with $\tilde{H}_c=\begin{pmatrix}H_{c_0} & H_{c_1} & \ldots & H_{c_n}\end{pmatrix}$ and $\Gamma=\begin{pmatrix}I\\ I\alpha_1\\ \vdots\\ I\alpha_n\end{pmatrix}$. The parameters α_i are time varying, and capture the time-varying properties of the communication network due to imperfect communication such as communication error, or communication failure. In essence, each α_i represents one possible failure, or source of error that the designer specifies as potentially occurring during operation. Given the problem under consideration, it is sufficient to consider $\alpha_i\geq 0\,\forall i$, however, negative values of α_i could alternatively be accounted for. Taking $\alpha_i\in[\underline{\alpha}_i,\bar{\alpha}_i]$, with $\underline{\alpha}_i<1<\bar{\alpha}_i$ allows for bounded noise in the controller communication network to be modeled (due to quantization error, for example). The case where $\underline{\alpha}_i=0$ allows for communication failure (either temporarily or permanently) to be modeled in ith communication channel. The topology represented by $H_c(\alpha)$ is a logical topology, that is, it captures where the communicated information leaves from and arrives. As such, it does not put any limitations on the physical topology of the network. The process network may be modeled in an analogous manner, however, in this article, the process network is taken to be constant. That is, the process network topology is captured by a constant matrix H_p, which typically has elements in the range $[0,1]$, representing no connection and partial or total connection. A matrix F_p is introduced to select which components of $\mathbf{y}(t)$ are interconnecting variables, thus, the process interconnection relations may be described as $u_p=H_pF_py$. The decomposition of the closed-loop process network is described in more detail later in the article.

Dissipativity and Dynamic Supply Rates

As an input–output property of systems, dissipativity is useful in studying interconnected systems as it allows for much of the complexity of the problem to be shifted to the interconnection relations, rather than studying centralized models. Once the dissipativity of the subsystems is ascertained, the dissipativity-based analysis for complex networks can be performed easily, yielding a scalable approach. A discrete-time dynamical system with input, output, and state $u\in\mathbb{R}^p, y\in\mathbb{R}^q$, and $x\in\mathbb{R}^n$, respectively, is said to be dissipative if there exists a function defined on the input and output variables, called the supply rate $s(u,y)$ and positive semidefinite function defined on the state, called the storage function $V(x(t))$ such that

$$V(x(t+1))-V(x(t))\leq s(u(t),y(t)) \tag{4}$$

for all time steps t.[27] The following (Q, S, R)-type of supply rate is commonly used

$$s(u(t),y(t))=y^T(t)Qy(t)+2y^T(t)Su(t)+u^T(t)Ru(t) \tag{5}$$

Quadratic differential forms were first introduced by Willems and Trentelman,[28] in the context of continuous time systems. This framework was then adapted to the discrete time case.[29] A QdF may be written in terms of extended inputs and outputs. Defining

$$\begin{aligned}\hat{u}^T(t)&=\begin{pmatrix}u^T(t) & u^T(t+1) & \ldots & u^T(t+\tilde{n})\end{pmatrix}\\ \hat{y}^T(t)&=\begin{pmatrix}y^T(t) & y^T(t+1) & \ldots & y^T(t+\tilde{m})\end{pmatrix}\end{aligned} \tag{6}$$

(for some finite $\tilde{n}$ and $\tilde{m}$), a QdF supply rate, denoted Q_ϕ, is defined as follows

$$Q_\phi(y,u)=\hat{y}^T(t)\tilde{Q}\hat{y}(t)+2\hat{y}^T(t)\tilde{S}\hat{u}(t)+\hat{u}^T(t)\tilde{R}\hat{u}(t) \tag{7}$$

$$Q_\phi(y,u)=\begin{pmatrix}\hat{y}(t)\\ \hat{u}(t)\end{pmatrix}^T\begin{pmatrix}\tilde{Q} & \tilde{S}\\ \tilde{S}^T & \tilde{R}\end{pmatrix}\begin{pmatrix}\hat{y}(t)\\ \hat{u}(t)\end{pmatrix} \tag{8}$$

Essentially, a QdF is a quadratic form similar to (5), although it is extended to include future inputs and outputs. This allows for a more detailed description of the system in terms of dissipativity, and fits in well with the MPC framework which is based around prediction. This additional system information allows for less conservative stability and performance results as compared to other dissipativity-based approaches.[26] Defining $\hat{w}(t)=[\hat{y}(t)^T,\hat{u}(t)^T]^T$, (8) can be written in a compact form as $Q_\phi(y,u)=\sum_{k=0}^{N}\sum_{l=0}^{N}\hat{w}^T(t+k)\phi_{kl}\hat{w}(t+l)$, where N is called the degree of supply rate, which is the maximum number of forward steps in the supply rate. In the supply rate given in (6)–(8), $N=\max\{\tilde{n},\tilde{m}\}$. Such a QdF is said to be induced by the symmetric two variable polynomial matrix $\phi(\zeta,\eta)$ defined as $\phi(\zeta,\eta)=\sum_{k=0}^{N}\sum_{l=0}^{N}\phi_{kl}\zeta^k\eta^l$.

Here, ϕ_{kl} is the (k,l)th coefficient matrix of $\phi(\zeta,\eta)$, and the indeterminates ζ and η represent a forward step in time on the left and right of $\phi(\zeta,\eta)$, respectively. The coefficient matrix of $\phi(\zeta,\eta)$, denoted $\tilde{\phi}$, is a (constant) matrix with (k,l)th block being ϕ_{kl}. For example, for the QdF supply rate in (8), we have $\tilde{\phi}=\begin{pmatrix}\tilde{Q} & \tilde{S}\\ \tilde{S}^T & \tilde{R}\end{pmatrix}$.

In this work, QdFs are used for both supply rates and storage functions. In this case (as in the behavioral systems theory[28]), a storage function is defined on the extended input and output. The dissipativity condition with a supply rate Q_ϕ and a storage function Q_ψ in the QdF can be represented as follows

$$\sum_{t=0}^{\infty}Q_\phi(w(t))\geq Q_\psi(w(t)) \tag{9}$$

Define ∇ as the rate of change of a QdF, $\nabla Q_\phi(w(t))=Q_\phi(w(t+1))-Q_\phi(w(t))$. A useful property of QdFs is that the rate of change of a QdF is itself a QdF, that is, $\nabla Q_\phi=Q_{\nabla\phi}$. That is, the rate of change of the QdF induced by $\phi(\zeta,\eta)$ is itself a QdF induced by $\nabla\phi(\zeta,\eta)$. This simplifies calculations as $\nabla\phi(\zeta,\eta)=(\zeta\eta-1)\phi(\zeta,\eta)$. Equation 9 may then be written as

$$Q_\phi(w(t))\geq Q_{\nabla\psi}(w(t)) \tag{10}$$

If $Q_\phi(w(t))>0\forall w(t)\neq 0$, then $\phi(\zeta,\eta)$ is said to be positive definite, with $\phi(\zeta,\eta)>0$. Note that $\tilde{\phi}>0\Rightarrow\phi(\zeta,\eta)>0$. Some results underlying dissipativity and its links to stability in this framework are briefly discussed below.

Theorem 1 (*Kojima and Takaba*[29]). *A discrete-time LTI system is asymptotically stable if there exists a symmetric two variable polynomial matrix $\psi(\zeta,\eta)\geq 0$ and $\nabla\psi<0$ for all input and output satisfying the system equations.*

Note that the asymptotic stability defined above is for the outputs, not states, of a system. To ensure asymptotic stability of the states the additional assumption of zero-state detectability is required, this is ensured for any minimal state realization of a linear system. The above theorem also provides a sufficient condition for the stability of nonlinear systems assuming zero-state detectability. Proposition 1, below, gives a linear matrix inequality (LMI) condition for determining the QdF dissipativity of a state-space model of discrete-time linear system.

Proposition 1 (*Tippett and Bao*[14]). *A discrete-time LTI system with state-space representation (A,B,C,D) is dissipative with the supply rate and storage function pair induced by $\phi(\zeta,\eta)$ and $\psi(\zeta,\eta)$, respectively, with the corresponding coefficient matrices $\tilde{\phi}$ and $\tilde{\psi}$ partitioned as $\tilde{\phi}=\begin{pmatrix}\tilde{\phi}_Q & \tilde{\phi}_S\\ \tilde{\phi}_S^{\,T} & \tilde{\phi}_R\end{pmatrix}$ and $\tilde{\psi}=\begin{pmatrix}\tilde{\psi}_X & \tilde{\psi}_Y\\ \tilde{\psi}_Y^{\,T} & \tilde{\psi}_Z\end{pmatrix}$, if and only if the following LMI is satisfied*

$$\begin{pmatrix}\mathbb{T}_{11} & \mathbb{T}_{12}\\ \mathbb{T}_{12}^T & \mathbb{T}_{22}\end{pmatrix}\geq 0 \tag{11}$$

with

$$\hat{C}=\begin{pmatrix}C\\ CA\\ \vdots\\ CA^N\end{pmatrix} \tag{12}$$

$$\hat{D}=\begin{pmatrix}D & 0 & \dots & 0 & 0\\ CB & D & \dots & 0 & 0\\ \vdots & \ddots & \ddots & \ddots & \vdots\\ CA^{N-1}B & CA^{N-2}B & \dots & CB & D\end{pmatrix}$$

$$\begin{aligned}\mathbb{T}_{11}&=\hat{C}^T[\tilde{\phi}_Q-\tilde{v}_X]\hat{C}\\ \mathbb{T}_{12}&=\hat{C}^T[\tilde{\phi}_Q-\tilde{v}_X]\hat{D}+\hat{C}^T[\tilde{\phi}_S-\tilde{v}_Y]\\ \mathbb{T}_{22}&=\hat{D}^T[\tilde{\phi}_Q-\tilde{v}_X]\hat{D}+\hat{D}^T[\tilde{\phi}_S-\tilde{v}_Y]\\ &\quad+[\tilde{\phi}_S-\tilde{v}_Y]^T\hat{D}+[\tilde{\phi}_R-\tilde{v}_Z]\end{aligned} \tag{13}$$

where N is the degree of the supply rate and $v(\zeta,\eta)=\nabla\psi(\zeta,\eta)$.

In the above result, there is no constraint on the definiteness of the storage function. In this work, we will require that the storage functions of the processes be positive semidefinite, $\psi(\zeta,\eta)\geq 0$, this may be ensured by solving Proposition 1 with the additional LMI constraint $\tilde{\psi}\geq 0$, which implies $\psi(\zeta,\eta)\geq 0$.

In this work, the dissipativity of the process units is represented based on the above definition, which describes the relation between the input and output spaces of a system. Given its model, the dissipativity of a process unit can be determined by solving the LMI in Proposition 1, with decision variables $\tilde{\phi}_Q,\tilde{\phi}_S,\tilde{\phi}_R,\tilde{v}_X,\tilde{v}_Y$, and $\tilde{v}_Z$. The dissipativity of the MPC controllers is defined as a constraint on their input–output trajectory as defined below.

Definition 1 (*Dissipative Trajectory*[14]). *A system (controller) is a said to trace a dissipative trajectory if, at all time instances k, the following dissipative trajectory inequality is satisfied*

$$W_k = \sum_{t=0}^{k} Q_{\phi_c}(u(t), y(t)) \geq 0 \quad (14)$$

where Q_{ϕ_c} *is the supply rate induced by* $\phi_c(\zeta,\eta)=\begin{pmatrix} Q_c(\zeta,\eta) & S_c(\zeta,\eta) \\ S_c^T(\zeta,\eta) & R_c(\zeta,\eta) \end{pmatrix}$.

The dissipative trajectory inequality ensures that the accumulated supply (W_k) of the controller is nonnegative. This is in contrast to the usual discrete-time dissipativity condition given in (9). The concept of a dissipative trajectory may be viewed as a slightly weaker version of the classic dissipativity as it does not explicitly require the dissipative trajectory inequality to be satisfied for all possible input $u(t)$. The MPC does not have a storage function, only a supply rate and an accumulation of supply. As such, if the controller has been "sufficiently dissipative" in the past, that is, more dissipation has occurred earlier in the trajectory than the minimum amount, then this accumulation of supply may be used as a storage to compensate for deficiencies in dissipation in the future.

Process Network Dissipativity Formulation

Dissipativity of individual subsystems

The chemical process network is decomposed into a network of process units interacting through physical (mass and energy) flows. The dissipativity of the process network is determined based on the dissipativity of each subsystem and their interconnections (this will be discussed in detail later). This network varies with changing recycle ratios and bypass fractions as well as larger structural changes to the network. Similarly, the distributed control system is represented as a network of controllers communicating through an information network. In this approach, the topologies of the process and controller networks are allowed to be arbitrary. In this article, we will restrict ourselves to the case where both networks have the same nominal topology to aid with online optimization. Adding additional links may improve the accuracy of the controllers predictions, however, at the cost of increased computational effort. As the number of decision variables is not changed by changing the controller communication network topology, it would be expected that this would not be very significant. Conversely, if the controller communication topology is completely different to that of the underlying process, the stability of the closed-loop process network will still be ensured by the dissipativity-based arguments to follow. However, the controller predictions, and as such, so to performance, may be degraded.

Consider a chemical process network with n process units. Partitioning the input to each process unit into the input from interconnected processes u_p, external disturbances d, and manipulated input u_c, the ith process unit P_i can be represented as in (1), and as shown in Figure 1a. Thus, if a QdF supply rate of the ith process is $Q_{\phi_i}(u_p, d, u_c, y)$, it is induced by a symmetric two variable polynomial matrix which may be conformally partitioned as

$$\phi_i(\zeta,\eta)=\begin{pmatrix} \mathcal{Q}_i(\zeta,\eta) & \mathcal{S}_i(\zeta,\eta) \\ \mathcal{S}_i^T(\zeta,\eta) & \mathcal{R}_i(\zeta,\eta) \end{pmatrix} \quad (15)$$

with

$$\mathcal{Q}_i(\zeta,\eta)=Q(\zeta,\eta)$$

$$\mathcal{S}_i(\zeta,\eta)=\begin{pmatrix} S_I(\zeta,\eta) & S_d(\zeta,\eta) & S_L(\zeta,\eta) \end{pmatrix}$$

$$\mathcal{R}_i(\zeta,\eta)=\begin{pmatrix} R_{II}(\zeta,\eta) & R_{Id}(\zeta,\eta) & R_{IL}(\zeta,\eta) \\ R_{Id}^T(\zeta,\eta) & R_{dd}(\zeta,\eta) & R_{dL}(\zeta,\eta) \\ R_{IL}^T(\zeta,\eta) & R_{dL}^T(\zeta,\eta) & R_{LL}(\zeta,\eta) \end{pmatrix}$$

As shown in the upper part of Figure 2a, process units are stacked diagonally to form the overall plant without interconnections, denoted as $\tilde{\mathcal{P}}$. The inputs and outputs of this system are the vectors consisting of the inputs and outputs of each system. For example, the output of this block diagonal system is $\mathbf{y}=(y_1^T \ldots y_n^T)^T$. The inputs $\mathbf{u_p}, \mathbf{d}$, and $\mathbf{u_c}$ are defined in the same way. Suppressing the dependence on ζ and η for convenience, the supply rate of this system is described as a QdF induced by the matrix

$$\Phi(\zeta,\eta)=\begin{pmatrix} \mathcal{Q} & \mathcal{S} \\ \mathcal{S}^T & \mathcal{R} \end{pmatrix} \quad (16)$$

with $\mathcal{Q}=\text{diag}(\mathcal{Q}_1,\ldots,\mathcal{Q}_i,\ldots,\mathcal{Q}_n)$ and $\mathcal{S}$ and $\mathcal{R}$ similarly defined. Knowing each $\phi_i(\zeta,\eta)(\theta_i(\zeta,\eta)), i=1,\ldots,n, \Phi(\zeta,\eta)$ $(\Theta(\zeta,\eta))$ can be easily calculated. The relation between the storage function of the ith process, induced by $\psi_i(\zeta,\eta)$, and storage function of $\tilde{\mathcal{P}}$, induced by $\Psi(\zeta,\eta)$, is completely analogous. The ability to easily formulate overall system dynamic features by considering only the features of the individual subsystems and interconnection relations is one feature of dissipativity-based analysis, which highlights its suitability in this context.

The distributed controller network is represented in a similar way. As shown in Figure 1b, each controller has two pairs of input and output: one set of local input (local sensor output) u_c and output (manipulated variable) y_L; one set of remote input (information received from other controllers) u_r and output (information sent to other controllers) y_r. Assume that the supply rate of the ith controller is induced by $\theta_i(\zeta,\eta)=\begin{pmatrix} \mathcal{Q}_{ci} & \mathcal{S}_{ci} \\ \mathcal{S}_{ci}^T & \mathcal{R}_{ci} \end{pmatrix}$. Then, $\mathcal{Q}_{ci}$ may be partitioned as $\begin{pmatrix} Q_{ci}^{LL} & Q_{ci}^{Lr} \\ Q_{ci}^{Lr^T} & Q_{ci}^{rr} \end{pmatrix}$, with $\mathcal{S}_{ci}$ and $\mathcal{R}_{ci}$ being analogously partitioned. The supply rate of the system composing of the diagonal stacking of all controllers, $\tilde{\mathcal{C}}$, is then induced by $\Theta(\zeta,\eta)=\begin{pmatrix} \mathcal{Q}_c & \mathcal{S}_c \\ \mathcal{S}_c^T & \mathcal{R}_c \end{pmatrix}$ analogously as for the process units.

Process network dissipativity formulation

With the structure of the supply rates and storage functions of each subsystem in the process network and the topology of the process and controller networks, the dissipativity properties of the process network may be determined. These will then be used in the following section to show stability of the process network. Figure 2b shows the process network with control with the varying parameters Γ (defined in Ref. 3) shown explicitly for analysis purposes. The varying parameters may then be "pulled" out and the system reformulated as shown in Figure 3. Where $\Delta(k)=\Gamma(\alpha)$, captures the time-varying parameters which model communication failure and noise. Using this formulation of the process network and the structure of the process and controller

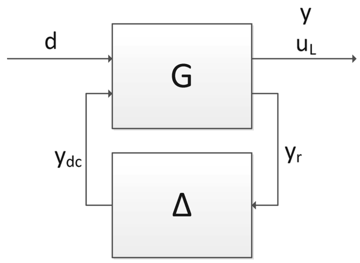


Figure 3. Reformulated process network.

supply rates developed in the previous section, the dissipativity of the process network with control may be developed. First, a condition for the dissipativity of the Δ system is presented.

Proposition 2. *The system Δ in Figure 3 is dissipative with supply rate induced by*

$$\phi_\Delta(\zeta,\eta)=\begin{pmatrix} Q_\Delta(\zeta,\eta) & S_\Delta(\zeta,\eta) \\ S_\Delta^T(\zeta,\eta) & R_\Delta(\zeta,\eta)\end{pmatrix} \tag{17}$$

if

$$\begin{pmatrix} P \\ I \end{pmatrix}^T \begin{pmatrix} Q_\Delta(\zeta,\eta) & S_\Delta(\zeta,\eta) \\ S_\Delta^T(\zeta,\eta) & R_\Delta(\zeta,\eta)\end{pmatrix}\begin{pmatrix} P \\ I \end{pmatrix} \geq 0 \tag{18}$$

where $P=\text{diag}_n\Gamma(\alpha), Q_\Delta(\zeta,\eta)=Q_\Gamma(\zeta,\eta)$ and $S_\Delta(\zeta,\eta), R_\Delta(\zeta,\eta)$ are similarly defined, and $\text{diag}_n A$ refers to the creation of a block diagonal matrix with n diagonal blocks A.

Proof. As $\Gamma(\alpha)$ is a memoryless system, the dissipativity inequality of the system Δ is $\sum^k Q_\Delta(k) \geq 0 \forall k$. Denoting the input–output space of Δ as (*u*,*y*) for brevity, this inequality in the extended input–output space is

$$\hat{y}^T Q_\Delta \hat{y} + 2\hat{y}^T S_\Delta \hat{u} + \hat{u}^T R_\Delta \hat{u} \geq 0 \tag{19}$$

Substituting in $\hat{y}=\text{diag}_n \Delta \hat{u}$ and realizing that (19) must hold for all $\hat{u}$ for Δ to be dissipative, the stated condition is achieved. ■

REMARK 1. *It should be noted that it is always possible to find a valid supply rate for a system. The difficulty is more about finding a tight dissipativity bound, which in the case that the parameters α representing the communication errors/failures vary by a wide margin may yield to looser bounds on the worst case performance and stability conditions.*

The supply rate in Proposition 2 may be found such that it is valid for all possible values α_i. Alternatively, different values of $Q_\Delta(\zeta,\eta), S_\Delta(\zeta,\eta)$, and $R_\Delta(\zeta,\eta)$ may be found to satisfy (18) for different values of α_i. The latter may lead to sharper results at the cost of a small increase in the complexity of the offline LMI problem to be presented later. The structure of the supply rate of the closed-loop process network may now be presented as a function of the supply rates of the individual processes, controllers, and the process and controller interconnection topology.

Proposition 3. *Consider the interconnected system with N processes and controllers as shown in Figure 22b, reformulated as in Figure 3. If the collection of processes, $\tilde{\mathcal{P}}$, is dissipative with respect to supply rate Q_ϕ and storage function Q_Ψ (with $\Psi(\zeta,\eta)\geq 0$), and the collection of controllers, $\tilde{\mathcal{C}}$, traces a dissipative trajectory with respect to supply rate Q_Θ, with Γ having supply rate Q_Γ. Then, the system from all disturbances **d** to all outputs $\mathbf{y_{pw}}=[\mathbf{y}^T, \mathbf{u_L}^T, \mathbf{y_r}^T]^T$ satisfies the following dissipativity condition*

$$\sum_{j=0}^{k} Q_\mu(j) \geq Q_\Psi(j) \geq 0 \tag{20}$$

for all time steps j, where

$$\mu(\zeta,\eta)=\begin{pmatrix} F_{11}(\zeta,\eta) & F_{12}(\zeta,\eta) \\ F_{12}^T(\zeta,\eta) & F_{22}(\zeta,\eta)\end{pmatrix} \tag{21}$$

$$F_{11}(\zeta,\eta)=\begin{pmatrix} F_{11} & F_{12} & F_{13} \\ \star & F_{22} & F_{23} \\ \star & \star & F_{33}\end{pmatrix} \tag{22}$$

$$F_{12}(\zeta,\eta)=\begin{pmatrix} \mathcal{S}_d+F_p^T H_p^T \mathcal{R}_{Id} \\ \mathcal{R}_{dL}^T \\ 0\end{pmatrix} \tag{23}$$

$$F_{22}(\zeta,\eta)=\mathcal{R}_{dd} \tag{24}$$

with

$$F_{11}=\mathcal{Q}+F_I^T \mathcal{R}_c^{II} F_I + \mathcal{S}_I H_p F_p + F_p^T H_p^T \mathcal{S}_I^T + F_p^T H_p^T \mathcal{R}_{II} H_p F_p \tag{25}$$

$$F_{12}=\mathcal{S}_L+F_I^T \mathcal{S}_c^{ll^T}+F_p^T H_p^T \mathcal{R}_{IL} \tag{26}$$

$$F_{13}=F_I^T \mathcal{S}_c^{lr}+F_I^T \mathcal{R}_c^{lr}\tilde{H}_c\Gamma(\alpha) \tag{27}$$

$$F_{22}=\mathcal{R}_{LL}+\mathcal{Q}_c^{ll} \tag{28}$$

$$F_{23}=\mathcal{Q}_c^{lr}+\mathcal{S}_c^{lr}\tilde{H}_c\Gamma(\alpha) \tag{29}$$

$$F_{33}=\mathcal{Q}_c^{rr}+\mathcal{R}_\Gamma+\mathcal{S}_c^{rr}\tilde{H}_c\Gamma(\alpha)+\Gamma^T(\alpha)\tilde{H}_c^T\mathcal{S}_c^{rr^T}+\Gamma^T(\alpha)\mathcal{S}_\Gamma + \mathcal{S}_\Gamma^T\Gamma(\alpha)+\Gamma^T(\alpha)\tilde{H}_c^T\mathcal{R}_c^{rr}\tilde{H}_c\Gamma(\alpha)+\Gamma^T(\alpha)\mathcal{Q}_\Gamma\Gamma(\alpha) \tag{30}$$

Proof. To improve readability, a sketch of the proof is presented. As discussed above, the supply rate and storage function of $\tilde{P}$ are induced by $\Phi(\zeta,\eta)$ and $\Psi(\zeta,\eta)$, respectively, similarly, the supply rate of $\tilde{C}$ is induced by $\Theta(\zeta,\eta)$. Then, the dissipativity of the system "*G*" in Figure 3 is built from the dissipativity inequalities of $\tilde{P}$ and $\tilde{C}$. That is, $Q_\Phi \geq Q_{\nabla\Psi}$ and $Q_\Theta \geq 0$, respectively. As well as the relations $\mathbf{y_L}=F_I\mathbf{y}, \mathbf{u_r}=\tilde{H}_c\mathbf{y_{dc}}$, and $\mathbf{u_p}=H_p\mathbf{y_{dp}}$ by summing the dissipation inequalities, substituting in the interconnection relations and rearranging. The supply rate of the Δ system may then be determined from Proposition 2. Summing the dissipation inequalities for *G* and Δ, and finally, substituting in the relation $\mathbf{y_{dc}}=\Gamma(\alpha)\mathbf{y}$, achieves the stated result. ■

Process network stability analysis

In the closed-loop process network dissipativity formulation presented in the previous section, the supply rate of the closed-loop process network is a function of the process and controller network topologies, and as such, will change with these networks. However, the storage function of the process network is a not a function of the α_i, instead it is a linear combination of the storage functions of the individual processes. We may then choose to select a single storage function for each process, in which case the process network will have a single storage function for all permutations of communication failures. Alternatively, however, we could allow each process to have a different storage function for each failure mode of the controller network. Choosing a single storage function is physically analogous to choosing a single energy function to describe the behavior of the closed-loop system regardless of the communication failures that occur. Choosing different storage functions may be interpreted as choosing different energy functions to describe the closed-loop networks behavior when different communication failures occur. Clearly, the latter approach may yield tighter results, although at the cost of increased complexity in the offline dissipativity assignment stage. The following results give conditions for the stability of the closed-loop process network in both of these cases assuming that the distributed MPC algorithm is recursively feasible (feasibility will be discussed in the following section). In the theorem below, the former case of a single storage function is considered; this has the advantage of improved scalability and flexibility as only a single storage function for each process is required.

Theorem 2. *Consider a process network with a storage function induced by* $\Psi(\zeta,\eta)\geq 0$, *with dissipative distributed MPCs, as shown in Figure 22. Assuming the MPC algorithm is initially and recursively feasible. If the system from external disturbances,* $\mathbf{d}$, *to combined process and controller outputs* $\mathbf{y_{pw}}=[\mathbf{y}^T,\mathbf{u_L}^T,\mathbf{y_r}^T]^T$ *traces a dissipative trajectory with respect to i switching supply rates* Q_{μ_i}, *and* $\mu_i(\zeta,\eta)=\begin{pmatrix}F_{11_i}(\zeta,\eta) & F_{12_i}(\zeta,\eta)\\ F_{12_i}^T(\zeta,\eta) & F_{22_i}(\zeta,\eta)\end{pmatrix}$ *with* $\tilde{F}_{11_i}<0\ \forall i$. *Then, the process network is asymptotically stable with minimum performance level*

$$||W\mathbf{y_{pw}}||_2\leq||\mathbf{d}||_2 \tag{31}$$

with $W(z)=\frac{1}{\sqrt[4]{2}\mathrm{d}(z)N(z)}$, *where* $\tilde{d}^T\tilde{d}\geq\max\left(\bar{\sigma}(\hat{F}_{22_i}-\hat{F}_{12_i}^T\hat{F}_{11_i}\hat{F}_{12_i}),\bar{\sigma}(\hat{F}_{12_i}^T\hat{F}_{11_i}\hat{F}_{12_i})\right)\forall i$ *and* $N(z)\geq\left(-\hat{F}_{11_i}(z)\right)^{\frac{1}{2}}\forall i$ *and* $\hat{\mu}=\begin{pmatrix}\hat{F}_{11} & \hat{F}_{12}\\ \hat{F}_{12}^T & \hat{F}_{22}\end{pmatrix}\geq\mu_i\forall i$.

Proof. We first show stability of the process network, followed by the norm bound. Note that $\tilde{F}_{11_i}<0\forall i\Rightarrow F_{11_i}(\zeta,\eta)\forall i$. The dissipativity of the process network implies that at any point in time, $k\geq 0$, when the system is in its ith mode

$$Q_{\mu_i}(k)\geq Q_{\nabla\Psi}(k) \tag{32}$$

For vanishing disturbances, as $F_{11_i}(\zeta,\eta)<0\forall i$, the dissipation inequality implies

$$0>Q_{\mu_i}(k)\geq Q_{\nabla\Psi}(k) \tag{33}$$

This combined with $\Psi(\zeta,\eta)\geq 0$ implies asymptotic stability of the process network as Q_Ψ acts as a common Lyapunov function for all failure modes of the process network (for vanishing external disturbance). This implies asymptotic stability of the outputs of the process network, assuming zero-state detectability ensures asymptotic stability of the states as well.

The condition $\hat{\mu}=\begin{pmatrix}\hat{F}_{11} & \hat{F}_{12}\\ \hat{F}_{12}^T & \hat{F}_{22}\end{pmatrix}\geq\mu_i\forall i$ implies that $\hat{\mu}$ is a valid supply rate for all of the modes of the system $\forall i$ as

$$Q_{\hat{\mu}}(j)\geq Q_{\mu_i}(j)\geq Q_{\nabla\Psi}(j),\quad\forall i,j \tag{34}$$

Such a $\hat{\mu}(\zeta,\eta)$ is guaranteed to exist as $\mu_i(\zeta,\eta)\in\mathbb{R}^{n\times n}(\zeta,\eta),\forall i$.

Thus

$$\sum_{j=0}^{k}Q_{\hat{\mu}}(j)\geq Q_\Psi(j),\quad\forall k\geq 0 \tag{35}$$

$$\sum_{j=0}^{k}Q_{\hat{\mu}}(j)\geq 0 \tag{36}$$

As $F_{11_i}(\zeta,\eta)<0\forall i$, it is always possible to find a $\hat{\mu}$ such that $\hat{F}_{11}(\zeta,\eta)<0$. Using the superscript ^ to denote extensions of the variables to include future values up to the order of the supply rates as in Proposition 1, the process network traces a dissipative trajectory implying

$$\sum_{i=0}^{k}\hat{\mathbf{y}}_{\mathbf{pw}}^T(k)\Gamma_{11}\hat{\mathbf{y}}_{\mathbf{pw}}(k)+2\hat{\mathbf{y}}_{\mathbf{pw}}^T(k)\Gamma_{12}\hat{\mathbf{d}}(k)+\hat{\mathbf{d}}^T(k)\Gamma_{22}\hat{\mathbf{d}}(k)\geq Q_\Psi(k+1)-Q_\Psi(0) \tag{37}$$

is satisfied for any disturbance. Assuming the system starts from rest $Q_\Psi(0)=0$, and given that $\Psi(\zeta,\eta)\geq 0$, we have

$$\sum_{i=0}^{k}\hat{\mathbf{y}}_{\mathbf{pw}}^T(k)\hat{F}_{11}\hat{\mathbf{y}}_{\mathbf{pw}}(k)+2\hat{\mathbf{y}}_{\mathbf{pw}}^T(k)\hat{F}_{12}\hat{\mathbf{d}}(k)+\hat{\mathbf{d}}^T(k)\hat{F}_{22}\hat{\mathbf{d}}(k)\geq 0 \tag{38}$$

Define $\bar{F}_{11}=-\hat{F}_{11}$, then

$$\sum_{i=0}^{k}\hat{\mathbf{y}}_{\mathbf{pw}}^T(k)\bar{F}_{11}\hat{\mathbf{y}}_{\mathbf{pw}}(k)-2\hat{\mathbf{y}}_{\mathbf{pw}}^T(k)\hat{F}_{12}\hat{\mathbf{d}}(k)\leq\hat{\mathbf{d}}^T(k)\hat{F}_{22}\hat{\mathbf{d}}(k) \tag{39}$$

Completing the square leads to

$$\sum_{i=0}^{k}\left[\bar{F}_{11}^{\frac{1}{2}}\hat{\mathbf{y}}_{\mathbf{pw}}-\bar{F}_{11}^{-\frac{1}{2}}\hat{F}_{12}\hat{\mathbf{d}}\right]^T\left[\bar{F}_{11}^{\frac{1}{2}}\hat{\mathbf{y}}_{\mathbf{pw}}-\bar{F}_{11}^{-\frac{1}{2}}\hat{F}_{12}\hat{\mathbf{d}}\right]\leq\sum_{i=0}^{k}\hat{\mathbf{d}}^T\left[\hat{F}_{22}+\hat{F}_{12}^T\bar{F}_{11}\hat{F}_{12}\right]\hat{\mathbf{d}} \tag{40}$$

Defining a row vector d of length $\deg(\hat{\mu}(\zeta,\eta))$ such that $d^Td\geq\max\left(\hat{F}_{22}+\hat{F}_{12}^T\bar{F}_{11}\hat{F}_{12},\hat{F}_{12}^T\bar{F}_{11}\hat{F}_{12}\right)$ and finally, using the reverse triangle inequality

$$||\bar{F}_{11}^{\frac{1}{2}}\hat{\mathbf{y}}_{\mathbf{pw}}||_2-||\bar{F}_{11}^{-\frac{1}{2}}\hat{F}_{12}\hat{\mathbf{d}}||_2\leq||d\hat{\mathbf{d}}||_2 \tag{41}$$

$$||\bar{F}_{11}^{\frac{1}{2}}\hat{\mathbf{y}}_{\mathbf{pw}}||_2\leq||\bar{F}_{11}^{-\frac{1}{2}}\hat{F}_{12}\hat{\mathbf{d}}||_2+||d\hat{\mathbf{d}}||_2 \tag{42}$$

$$||\bar{F}_{11}^{\frac{1}{2}}\hat{\mathbf{y}}_{\mathbf{pw}}||_2\leq 2||d\hat{\mathbf{d}}||_2 \tag{43}$$

which in the original variables, is

$$||\bar{F}_{11}^{\frac{1}{2}}(\eta)\mathbf{y}_{\mathbf{pw}}||_2\leq 2||d(\eta)\mathbf{d}||_2 \tag{44}$$

$$\left\|\frac{\bar{F}_{11}^{\frac{1}{2}}(\eta)}{\sqrt[4]{2}\mathrm{d}(\eta)\mathbf{y}_{\mathbf{pw}}}\right\|_2\leq||\mathbf{d}||_2 \tag{45}$$

where in the last line the fact that $d(\eta)$ is a scalar is used. ■

The result below is given for completeness, and shows stability for the case that the storage function of one or more processes changes if and when the communication network changes. In this case, there is implicitly a minimum dwell time between changes/failures in the network enforced by (46).

Theorem 3. *Consider a process network with storage functions induced by* $\Psi_i(\zeta,\eta) \geq 0$ *(for all i failure modes) with dissipative distributed MPCs. Assume the MPC algorithm is initially and recursively feasible. If the system from external disturbances,* $\mathbf{d}$*, to combined process and controller outputs* $\mathbf{y_{pw}}=[\mathbf{y}^T,\mathbf{u_L}^T,\mathbf{y_r}^T]^T$ *traces a dissipative trajectory with respect to i switching supply rates* Q_{μ_i}*, and* $\mu_i(\zeta,\eta)=\begin{pmatrix} F_{11_i}(\zeta,\eta) & F_{12_i}(\zeta,\eta) \\ F_{12_i}^T(\zeta,\eta) & F_{22_i}(\zeta,\eta)\end{pmatrix}$ *with* $\tilde{F}_{11_i} < 0 \forall i$*. If the modes change sufficiently slowly such that the value of the ith storage function at the time step that the system changes from its ith mode for the fth time,* $Q_{\Psi_i}(k)$*, and the value of the ith storage function the time step the process network changes into the ith mode for the* $(f+1)$*th time,* $Q_{\Psi_i}(j)(j>k)$*, satisfy*

$$Q_{\Psi_i}(k) \geq Q_{\Psi_i}(j) \tag{46}$$

Then, the process network is asymptotically stable with minimum performance level

$$||W\mathbf{y_{pw}}||_2 \leq ||\mathbf{d}||_2 \tag{47}$$

with $W(z)=\frac{1}{d(z)}N(z)$*, where* $\tilde{d}^T\tilde{d} \geq \max(\bar{\sigma}(\hat{F}_{22_i}-\hat{F}_{12_i}^T\hat{F}_{11_i}\hat{F}_{12_i}), \bar{\sigma}(\hat{F}_{12_i}^T\hat{F}_{11_i}\hat{F}_{12_i}))\forall i, N(z) \geq \left(-\hat{F}_{11_i}(z)\right)^{\frac{1}{2}}\forall i$*, and* $\hat{\mu}=\begin{pmatrix}\hat{F}_{11} & \hat{F}_{12} \\ \hat{F}_{12}^T & \hat{F}_{22}\end{pmatrix} \geq \mu_i \forall i$*.*

Proof. Condition (46) ensures that the ith storage function does not increase from its value in the last time step for the previous period the ith system was active (time $t=k$) when the communication network switches back into the ith mode the next time (time $t=j$). Using the same arguments as in Theorem 2, $\tilde{F}_{11_i} < 0 \Rightarrow F_{11_i}(\zeta,\eta) < 0 \forall i$ implies that (for vanishing external disturbance) when the ith storage function is active, it is strictly decreasing so long as $\mathbf{y_{pw}} \neq 0$. Thus, they act as multiple Lyapunov functions (as $\Psi_i(\zeta,\eta) \geq 0 \forall i$) ensuring asymptotic stability of the process network. The gain condition may be proved along the same lines as in Theorem 2. ■

The norm bounds presented above are worst case performance bounds; the achieved performance depends on rate of change of the parameter α as well as how close to optimal the control law is. For example, if $\alpha=\alpha_i$ is constant (indefinitely, or for a long period of time), then a valid supply rate is induced by $\mu_i(\zeta,\eta)$ (for a certain period of time at least), which, due to the nature of the construction of $\hat{\mu}(\zeta,\eta)$ above, will lead to a tighter norm bound. The stated norm bound will be achieved if the closed-loop process network is losslessly dissipative, if the network is not lossless (i.e., the dissipation rate is nonzero) then the actual performance may be better than this lower bound, even in the case of changing α.

Distributed MPC with Varying Communication Network

In this section, conditions are developed to ensure stability in the case that the parameter α_i is changing and not known. This "robust" approach is developed in the case of unknown changes in the controller communication network, the yield a PCS condition. Results for unknown changes in the process network topology may be developed in a completely analogous manner; this has been omitted, however, due to space constraints. The offline dissipativity assignment problem is first stated so as to ensure recursive feasibility, stability, and performance of the process network. Subsequently, the online optimization algorithm is presented along with conditions for the recursive feasibility of the optimization problem.

Offline dissipativity-based planning

In this section, we present an LMI optimization problem that is solved offline to determine the supply rates that the controllers must satisfy to ensure stability and minimum performance specifications of the process network. A single set of controller dissipativity properties are found to control the closed-loop process network for unknown changes α_i. The set $\mathcal{A}$ may have either finite or infinite cardinality, and the problem below need only be solved on the vertices of $\mathcal{A}$. This then implies that the condition holds true for all $\alpha_i \in \mathcal{A}$. This latter case allows for the plant-wide connectivity stability condition mentioned earlier.

The following problem is solved offline to assign the required dissipativity properties to each controller. This is done by finding a valid set of dissipativity properties for the individual processes, controllers, and of the Δ system such that the closed-loop process network is stable and achieves the desired minimum performance bounds. This problem is solved on the vertices of the set $\mathcal{A}$, which represent the extremal values of the α_i as specified by the designer. These are determined by which channels are prone to failure and the $\mathcal{L}_\infty$-norm of the noise/error in communication, which is specified by the designer *a priori*. Similar to $\mathcal{H}_\infty$ control approaches, if a solution does not exist to this problem, then the desired level of worst case performance or robustness to communication imperfections, must be decreased.

Problem 1. *Consider a process network with n individual processes and controllers with process and controller network topologies defined by* H_p *and* $H_c(\alpha)$*. Given* $\psi_i \geq 0 \forall i \in [1,n]$*, the cost functions of the ith controller. Find a set of supply rates parameterized by* $Q_{il}, S_{il}, R_i, Q_{c_i}, S_{c_i}$*, and* R_{c_i} *such that* ψ_i *is the storage function of the ith process, and the following LMI constraints are satisfied for all required values of* α_k *(for possibly different values of the process and controller supply rates).*

$$\begin{pmatrix}\mathbb{T}_{11_{il}} & \mathbb{T}_{12_{il}} \\ \mathbb{T}_{12_{il}}^T & \mathbb{T}_{22_{il}}\end{pmatrix} \geq 0 \ \forall i,l \tag{48}$$

$$(\textit{Dissipativity of ith process})$$

$$\begin{pmatrix} P \\ I \end{pmatrix}^T \begin{pmatrix} Q_\Delta(\zeta,\eta) & S_\Delta(\zeta,\eta) \\ S_\Delta^T(\zeta,\eta) & R_\Delta(\zeta,\eta)\end{pmatrix}\begin{pmatrix} P \\ I \end{pmatrix} \geq 0 \tag{49}$$

$$(\textit{Dissipativity of } \Delta \textit{ system})$$

$$\Omega_{i_l} \geq 0 \ \forall i,l \tag{50}$$

$$\chi_{i_l} < 0 \ \forall i \tag{51}$$

$$l\mathcal{Q}_{\Gamma_l} + \tilde{H}_c^T \mathcal{R}_{c_l}^{rr} \tilde{H}_c \leq 0 \ \forall l \tag{52}$$

$$(Feasibility\ of\ ith\ controller\ supply\ rate)$$

$$\tilde{F}_{11_l}^T < 0 \ \forall l \tag{53}$$

$$(Internal\ stability\ of\ process\ network\ with\ control)$$

$$\tilde{F}_{11_l} \leq -\tilde{N}_l^T \tilde{N}_l \ \forall l \tag{54}$$

$$\tilde{F}_{12_l} = 0 \ \forall l \tag{55}$$

$$\tilde{F}_{22_l} \geq \tilde{d}_l^T \tilde{d}_l \ \forall l \tag{56}$$

$$(Minimum\ performance\ guarantee)$$

The first LMI condition ensures the dissipativity of the ith process for the l values of α_i required as per Proposition 1. Using Proposition 2, Condition (49) implies that the varying or uncertain elements of the process and controller networks are suitably dissipative. Conditions (50)–(52) ensure that the controller supply rate is assured to be feasible and that the controllers are ensured to be dissipative for all possible values of α_i. Condition (52) is required as the terms $\Gamma^T(\alpha)\left(\mathcal{Q}_{\Gamma_l} + \tilde{H}_c^T \mathcal{R}_{c_l}^{rr} \tilde{H}_c\right)\Gamma(\alpha)$ in the supply rate of the closed-loop process network (formulated in Proposition 3) may not be convex in α. Setting $\mathcal{Q}_{\Gamma_l} + \tilde{H}_c^T \mathcal{R}_{c_l}^{rr} \tilde{H}_c \leq 0 \forall l$, however, ensures that whatever the value of α_i this term only helps ensure stability (by assisting in ensuring that $\tilde{F}_{11} < 0$). The internal stability constraint is required such that the results in Theorems 2 and 3 may be used to show asymptotic stability of the process network.

The final three conditions in the above problem allow the designer to specify a minimum performance level for the process network that is assured to be met. Note that $\tilde{F}_{11_l} \leq -\tilde{N}_l^T \tilde{N}_l, \forall l$ is not sufficient to show stability using Theorems 2 and 3, as in general it alone ensures $F_{11_l}(\zeta, \eta) \leq 0, \forall l$ (although $N(\xi)$ may be chosen by the designer to ensure the strict inequality). As such, Condition (53) is implemented to ensure internal stability of the closed loop process network regardless of choice of weighting function. If, however, $\tilde{N}_l^T \tilde{N}_l < 0, \forall l$ then Condition (53) is automatically satisfied and may be removed.

In the following, conditions for the stability of the process network in the case of failures in the controller communication network are developed. A similar "PCS" condition has recently been presented[15] for the case of (Q, S, R)-type supply rates for linear distributed controllers. The framework presented in the current article is more general than this existing result, as structured failures of the controller network can be considered.

That is, if some communication links are more prone to failure (e.g., wireless links) then stability in the case of these failures only can be ensured, whereas, using the approach in Ref. 15 only stability in the case of all permutations of possible failures in the controller network can be ensured. Thus, leading to possibly conservative results in cases where some links are unlikely to fail. The following result can be used to ensure stability in the case of structured or arbitrary (as determined by Eq. 2) failures in the controller communication network. Note that in this case the controllers do not have knowledge of the value of the α_i.

Theorem 4 (*PCS*). *The process network with distributed dissipative MPC is internally asymptotically stable in the presence of variable controller communication network parameterized by (2), where all $\alpha_i \in \mathcal{A}$. If, all controllers trace a dissipative trajectory such that $\tilde{F}_{11} < 0$ for the vertices of $\mathcal{A}$ and $\tilde{H}_c^T \mathcal{R}_c^{rr} \tilde{H}_c + \mathcal{Q}_\Gamma \leq 0$, with a single supply rate for each individual controller, and the individual processes have nonnegative storage functions.*

Proof. The proof is given for the case that there is a single (nonswitching) storage function for each process, although stability may be shown for multiple storage functions as in Theorem 3, although assumptions on the minimum dwell time are required. If $\tilde{F}_{11} < 0$ for the vertices of $\mathcal{A}$ and $\tilde{H}_c^T \mathcal{R}_c^{rr} \tilde{H}_c + \mathcal{Q}_\Gamma \leq 0$, then it is well known that $\tilde{F}_{11} < 0$ for all $\alpha_i \in \mathcal{A}$.[30] The assumption that the individual controller and process supply rates are such that process network is dissipative with supply rate induced by $\mu(\zeta,\eta) = \begin{pmatrix} F_{11}(\zeta,\eta) & F_{12}(\zeta,\eta) \\ F_{12}^T(\zeta,\eta) & F_{22}(\zeta,\eta) \end{pmatrix}$ with $\tilde{F}_{11} < 0$ implies that $Q_{\nabla\Psi} < 0$ for vanishing disturbance. This, combined with $\psi_i(\zeta,\eta) \geq 0$ for all i implies that $\Psi(\zeta,\eta) \geq 0$, thus implying asymptotic stability with Q_Ψ acting as a Lyapunov function. ■

In the above result, there is no restriction on the rate at which the α_i change. As such, in addition to modeling failure in the controller communication network, if $\underline{\alpha}_i \leq 1 \leq \bar{\alpha}_i$, then this framework can be used to model noise (due to quantization error, for example) in the controller communication network.

Online dissipative MPC algorithm

The online dissipative distributed MPC algorithm is to a large extent the same as the algorithm presented in our previous work[14]; the key novelty of the current article is the offline dissipativity-based planning problem, which differs substantially from our previous work. However, a point of difference of the online problem in the current article is that the communication may break down between controllers. In which case, the terms in the ith controllers supply rate associated with these signals disappear as the input from the remote controller(s) is nonexistent. This is due to the connection between the controllers being severed (at least temporarily), this is consistent with existing fault tolerant control approaches based on passivity.[31] It should be noted that in the current article, as in the authors previous work,[14] the controllers communicate by sending their predictions of their own local processes future output, which are in turn used by the other controllers to improve their own local process predictions and, therefore, to help optimize their local control action. When communication breaks down, the controllers may use the previously received trajectory information to help compute their local control action, however, as mentioned above, this is not used in the dissipative trajectory constraint and, as such, does not effect plant-wide stability. The online dissipative distributed MPC algorithm is briefly described in the following for completeness. The underlying problem that is to be solved is:

Problem 2. *For every time k solve*

$$\hat{u}_c(k) = \operatorname{argmin} \sum_{i=0}^{N} y^T(k) Q y(k) + u_c^T(k) R u_c(k) + \epsilon, \tag{57}$$

subject to

$$\hat{u}_c(k) \in \mathcal{U} \tag{58}$$

$$\hat{y}^T(k) P \hat{y}(k) \leq 1 + \epsilon \tag{59}$$

$$P_L : \left\{ \begin{pmatrix} x(k+1) \\ y(k) \end{pmatrix} = \begin{pmatrix} A & B_1 & B_2 & B_3 \\ C & D_1 & D_2 & D_3 \end{pmatrix} \begin{pmatrix} x(k) \\ u_c(k) \\ u_p(k) \\ d(k) \end{pmatrix} \right. \tag{60}$$

$$Q_{\phi_c}(u_L(k), u_r(k), u_c(k), y_r(k)) + W_{k-1} \geq 0 \tag{61}$$

In the above, $\mathcal{U}$ a convex set, this allows for hard or soft convex input constraints such as ellipsoidal or box constraints to be implemented. Problem 2 is a quadratic program with a quadratic constraint. In general, these are nonconvex, however, $\phi_{c_i}(\zeta, \eta)$ may be chosen such that the above problem is transformed into an LMI problem. A description of this is provided in the appendix, along with a condition for the feasibility of the MPC algorithm in the presence of hard input constraints. The reader directed to Ref. 14 for a full description.

Alternatively, recursive feasibility may be assured if the MPC algorithm is at least as dissipative as a known, linear observable, distributed controller which satisfies the required dissipativity properties for stability and performance. In the presence of hard input constraints, it is also necessary that this controller satisfies them for the desired initial conditions and disturbances. Essentially, the dissipation of the MPC is compared to that of the known controller each step, and if it is less dissipative than the known controller, then the output of the known controller is applied. This is made precise below.

Proposition 4. *If there exists a known, linear observable, distributed controller which is dissipative with the same supply rate as the MPC, Q_Θ, and storage function $Q_\Sigma \geq 0$. Then, recursive feasibility may be assured for the softly constrained case, if, at all time steps the MPC algorithm satisfies*

- *If the MPC algorithm satisfies $Q_\Theta - Q_{\nabla\Sigma} \geq 0$ then, apply the result of the optimization problem,*
- *Otherwise, apply the output of the distributed controller.*

Proof. Clearly the MPC satisfies the difference form of the known distributed controller dissipation inequality $Q_\Theta \geq Q_{\nabla\Sigma}$, for both of the above cases. Summing, this dissipation inequality and using $Q_\Sigma \geq 0$ this in turn implies that the controller dissipative trajectory inequality, $\sum_{i=0}^{k} Q_\Theta(i) \geq 0$, is satisfied $\forall k$. As the distributed controller is assumed to be linear observable, the existence of a solution to the known controller output based on a finite number of the past inputs and outputs (and potentially the current input) is guaranteed. ■

This approach may be thought of as an adaption of the Lyapunov MPC approach developed by Christofides and coworkers,[32] where the distributed MPC is required to contract a Lyapunov function at least as much as a known, stabilizing, centralized controller, otherwise the output of the centralized controller is used. A disadvantage of Proposition 4 as compared to that presented in Proposition 5, is that knowledge of a stabilizing distributed controller is required, although it should be noted that this does not add to the complexity of the online problem. It has the advantage, however, of possibly being less conservative.

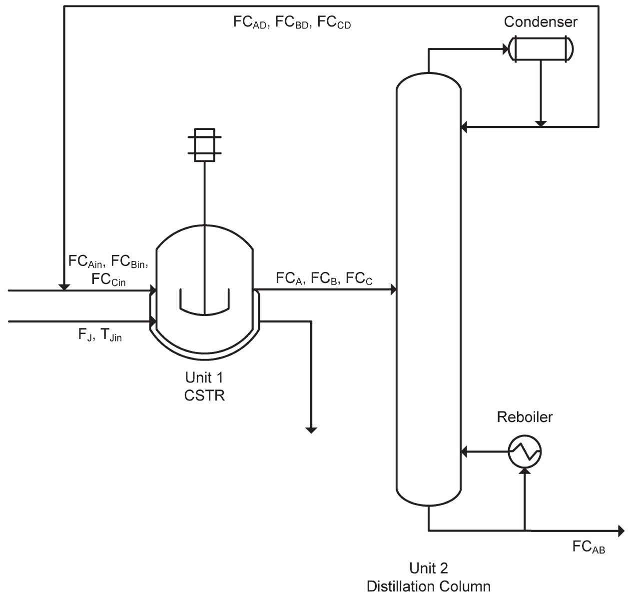


Figure 4. Process network with recycle and purge.

Illustrative Example

To improve transparency as an illustration of the technical results in this article, we consider the process network shown in Figure 4, consisting of a continuously stirred tank reactor (CSTR) and distillation column in a recycle arrangement. The irreversible reaction $A \rightarrow B$ occurs in the reactor, the fresh feed to the CSTR consists of A and C, an inert component. C is the light component of the mixture, and as such is concentrated in the distillate product of the column. As such, a purge is used to prevent it from building up.

Using the network description presented earlier in the article, the process network may be represented by

$$H_p(\alpha) = \begin{pmatrix} 0 & 0.95I \\ I & 0 \end{pmatrix} \tag{62}$$

Note that in this case H_p is a constant matrix, as it is assumed that it does not vary in an unknown manner, although the more general form of the approach developed in this article allows for it. A distributed control system is to be designed so as to produce the desired product in the face of possible failure of the controller communication network, that is, subject to the PCS condition in Theorem 4. The controller communication network is assumed to mimic the process network in underlying structure. Thus, the controller network may be represented as

$$H_c(\alpha) = \begin{pmatrix} 0 & \alpha_1 I \\ \alpha_2 I & 0 \end{pmatrix} \tag{63}$$

with $\alpha_1 \in \{0, 1\}$ and $\alpha_2 \in \{0, 1\}$, where $\alpha_i = 0$ implies no communication (communication failure) and $\alpha_i = 1$ implies communication is occurring. The linearized discrete-time models of the CSTR and distillation column are

$$\begin{pmatrix} A_c & B_c \\ C_c & D_c \end{pmatrix} = \left(\begin{array}{ccccc|ccccc} -0.2859 & 0 & 0 & -0.1086 & -0.0101 & 0.7449 & 0 & 0 & 0.0229 & 0.0229 \\ 0.8490 & 0.5599 & 0 & 0.1074 & 0.01 & -1.0256 & 1.5176 & 0 & -0.0228 & -0.0228 \\ 0 & 0 & 0.5599 & 0 & 0 & 0 & 0 & 1.5176 & 0 & 0 \\ 5.6171 & 0 & 0 & 0.8724 & 0.0884 & -0.5693 & 0 & 0 & -0.3532 & -0.3532 \\ 0.2614 & 0 & 0 & 0.0442 & 0.0045 & -0.0378 & 0 & 0 & -0.3847 & -0.3847 \\ 0.29 & 0 & 0 & 0 & 0 & 0.34 & 0 & 0 & 0 & 0 \\ 0 & 0.29 & 0 & 0 & 0 & 0.85 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.29 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right) \tag{64}$$

$$\begin{pmatrix} A_d & B_d \\ C_d & D_d \end{pmatrix} = \left(\begin{array}{cccccc|ccccc} 0.0067 & 0.0164 & 0 & 0 & 0 & 0 & 0.0169 & 0.0169 & 0.0169 & 0.0169 & -0.0065 \\ -0.0067 & -0.0091 & 0 & 0 & 0 & 0 & 0.0046 & 0.0046 & 0.0046 & 0.0046 & -0.0105 \\ 0.0076 & 0 & 0.0152 & 0.0354 & 0.0478 & 0.0259 & 0.1326 & -0.0435 & 0.0833 & 0.0266 & -0.0158 \\ 0.0038 & -0.0022 & 0.0129 & 0.0329 & 0.0519 & 0.0307 & 0.034 & -0.0167 & 0.0198 & 0.0288 & -0.0199 \\ 0.003 & -0.0042 & 0.0195 & 0.0582 & 0.1137 & 0.0756 & -0.0001 & -0.0257 & -0.0073 & 0.0268 & -0.0507 \\ 0.0014 & -0.0027 & 0.0125 & 0.041 & 0.0898 & 0.0666 & -0.0365 & -0.0465 & -0.0393 & -0.098 & -0.0578 \\ 1 & 0 & 0 & 0 & 0 & 0 & & & & & \\ 0 & 1 & 0 & 0 & 0 & 0 & & & 0_{4\times 5} & & \\ 0 & 0 & 0 & 0 & 0 & 1 & & & & & \\ 0 & 0 & 0 & 0 & 0 & -1 & & & & & \end{array}\right) \tag{65}$$

respectively. The models for the CSTR and distillation column are taken from Refs. 33 and 34, respectively, and assume constant holdup volume. As per our network decomposition, partition the inputs to the reactor as $u_{\text{CSTR}}^T = \left(u_P^T \mid d^T \mid u_L^T\right)^T = (FC_{A_{\text{in}}} FC_{B_{\text{in}}} FC_{C_{\text{in}}} \mid T_{J_{\text{in}}} \mid F_J)$. That is, the interconnecting inputs are the molar flow rates of A, B, and C. The disturbance consists of a variation in the inlet cooling water temperature ($T_{J_{\text{in}}}$). Finally, the manipulated variable (local controller output, u_L) is the flow rate of water in the jacket, F_J. The inputs to the distillation column may be partitioned as

$$u_{\text{DC}}^T = \left(u_P^T \mid u_L^T\right) = (FC_A FC_B FC_C \mid C_{\text{v}} T_{\text{s}}) \tag{66}$$

That is, the interconnecting inputs, u_P, are the molar flow rates of A and B (in the outflow of the reactor). The

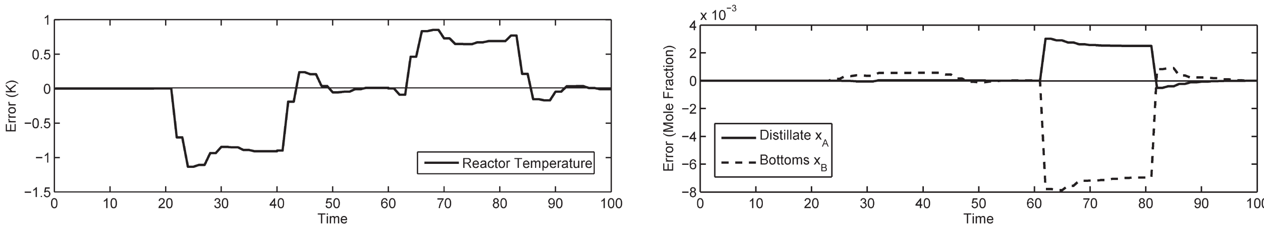


Figure 5. Controlled variables of the process units under distributed control, Case 1.

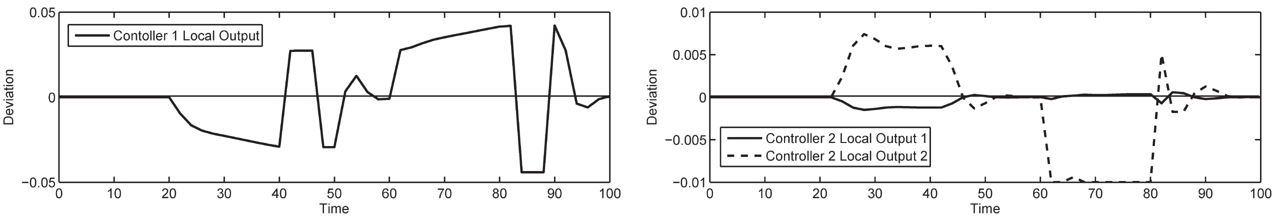


Figure 6. Distributed controller local outputs, Case 1.

manipulated variables (local controller outputs), u_L, are the reflux flow rate (by manipulating a valve position, C_v) and temperature of steam to the reboiler (T_s), respectively. The outputs are the molar flow rates of A, B, and C in the distillate (an interconnecting flow to the reactor) and the molar flow rate of A in the bottoms product, that is, $y_{DC}^T = (FC_{A_D} \quad FC_{B_D} \quad FC_{C_D} \quad FC_{A_B})$. It is assumed that the measured outputs of the distillation column (inputs to the controller) are the molar flow rates of A in the distillate and bottoms product.

Two distributed model predictive controllers are designed using the aforementioned methodology. The local output of the controller for the CSTR (Controller 1) are constrained to lie in $-1 < u_{L_1} < 1$, and the local output of the controller for the Distillation Column (Controller 2) are constrained to lie in $-0.01 < u_{L_2} < 0.01$. In the first case (Case 1), the distributed control system is designed to be robust with respect to all possible failures in the controller communication network, by using Theorem 4 developed above. That is, $\alpha_1^1 \in \{0, 1\}$ and $\alpha_2^1 \in \{0, 1\}$. In the second case (Case 2), the distributed controller is designed to be robust with respect to failure in the communication from Controller 1 to Controller 2 only, that is, $\alpha_1^2 \in \{0, 1\}$ and $\alpha_2^2 \in \{1\}$. The process network is designed to operate using a feedstock containing 5% C.

Figures 5 and 6 show the results of a simulation of the process network with distributed controller in Case 1. All simulations were carried out with a pulse disturbance in the CSTR jacket inlet temperature of magnitude $2K$ for $t = 20$–40, and a pulse disturbance in the inlet mole fraction of A of magnitude 0.15 from $t=60$ to 80. To illustrate the PCS condition developed in Theorem 4, the communication from Controller 1 to Controller 2 is simulated as failing at $t=70$. The simulations for Case 2 are shown in Figures 7 and 8. It can be seen that the performance in Case 2 is better than in Case 1, which is most evident in the comparison of the CSTR controlled variable. The integral time averaged error (ITAE) for the controlled variables in Case 1 was 2.01×10^3, compared to 1.09×10^3 in Case 2, this corresponds to a 45.8% decrease in the ITAE for Case 2. This improvement in performance is due to Case 2 being less conservative as

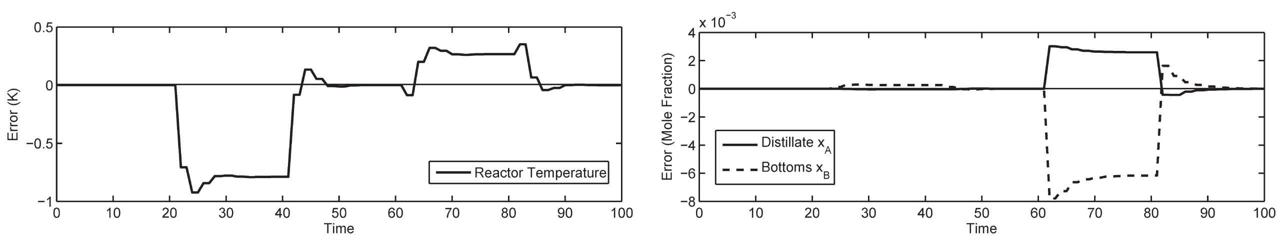


Figure 7. Controlled variables of the process units under distributed control, Case 2.

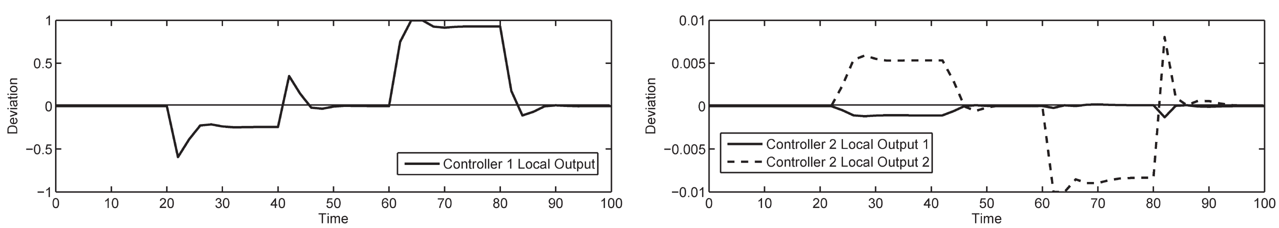


Figure 8. Distributed controller local outputs, Case 2.

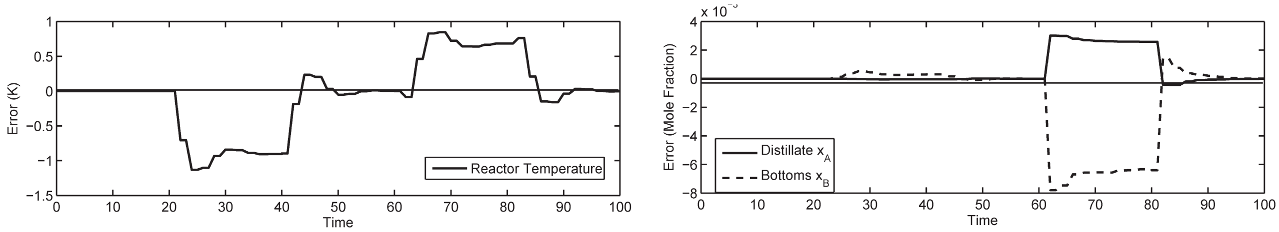


Figure 9. Controlled variables of the process units under decentralized control.

stability is only ensured for partial failure of the controller communication network, which in these simulations is all that is necessary.

To show the generality of the proposed approach, these results may be compared to the decentralized MPC case, which is shown in Figures 9 and 10. The decentralized controllers were designed in an analogous manner to the above distributed approach. In this case, however, the PCS condition is irrelevant as there is no controller communication, and H_c is a matrix of zeros. On comparing the results of the simulations, it is clear that the control performance is better in the distributed case, which also has more aggressive control action, this is especially clear in the CSTR controlled and manipulated variables.

An explanation for this improvement is that the communication between controllers improves predictions. Also, the communication may allow controllers to trade performance and stability with one another through the controller communication network. It should be noted, however, that if PCS is ensured for arbitrary controller network failure (which is the case in this example), then communication is only used to improve performance, not to assist in ensuring stability of the closed loop process network.

In Figure 11, the accumulated supply of the distributed (Case 1) and decentralized controllers are shown. It is clear to see that all controllers trace a dissipative trajectory as the accumulated supply is nonnegative at all times. It is interesting to note that the accumulated supply of the distributed controller is nondecreasing, unlike that of the decentralized controller. This may imply that the dissipative trajectory condition is easier to satisfy in the distributed case as compared to the decentralized case, and as such the distributed controller does not need to use its accumulated supply to augment its dissipativity at any time steps.

Discussion and Conclusions

An advantage of the proposed distributed MPC approach over a centralized approach is the reduced design complexity as control design is based on models of individual process units rather than complex plant-wide models process models. As each controller in the controller network is autonomous, this decentralization of the decision making process can lead to a more fault tolerant control system. In the proposed distributed control structure, although communication is carried out in an iterative manner, stability is ensured even if the iteration is halted after an arbitrary number of iterates (including one). As such computation can be highly parallelized. This advantage is expected to be more pronounced when nonlinear models are considered, due to the increased complexity of the optimization problem.

Results are presented ensuring stability of the closed-loop process network in the case of failure (complete or partial) of the controller communication network, and in the presence of bounded communication error (i.e., due to quantization error).

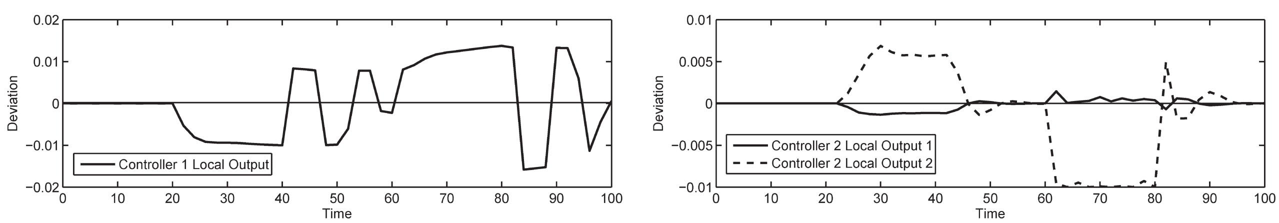
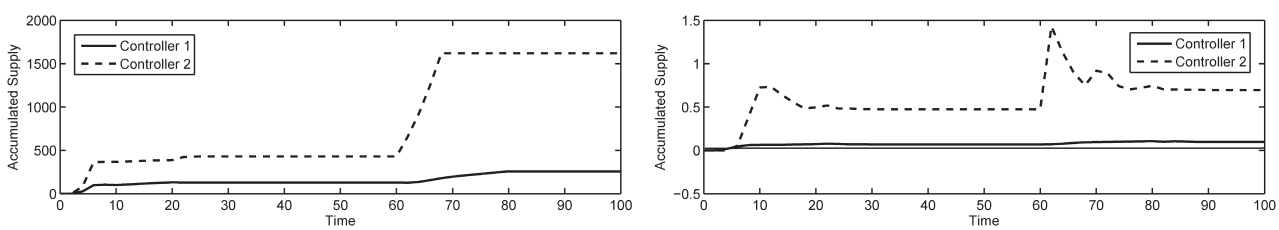


Figure 10. Decentralized controller local outputs.



(a) (b)

Figure 11. Accumulated supply of dissipative model predictive controllers.

(a) Distributed controllers, Case 1. (b) Decentralized controllers.

This can be seen as a more general form of the "PCS" condition developed in Ref. 15, which used (Q, S, R)-type supply rates and closed form controllers. An advantage of the framework presented in this article is that specific structured failures in the communication network can be studied (as opposed to ensuring stability for all possible permutations of failures). This finds relevance in practice when some communication links are wireless, and thus, more prone to failure than wired links; as may be the case in geographically distributed systems. The proposed approach facilitates a scalable and systematic approach to analyzing the effect of communication failure on plant-wide stability and worst-case performance.

Stability and minimum performance of the process network is ensured by requiring the controllers to have certain dissipativity properties such that the dissipativity of the process network takes on a particular form. This presents a scalable approach to control of process networks, as the dissipativity properties of process networks are a linear combination of the dissipativity properties of the individual process and controller networks regardless of the topology of these networks. The use of QdFs as supply rates allows for less conservative conditions as compared to traditional (Q, S, R)-type supply rates. Recursive feasibility of the dissipative MPC algorithm is ensured offline concurrently with the determination of the required controller dissipativity properties in an LMI optimization problem. As a potential generalization of the approach developed in this article, a unified framework for dissipative distributed MPC for changing process and controller network topologies could be developed. In the case of known changes in the network topologies, the individual controllers may be designed such that they reconfigure their dissipativity properties (essentially their supply rates) to account for, and possibly take advantage of, any changes in the process (and controller) network properties. To treat this problem, the process network can be parameterized in an analogous manner to the controller communication network, that is

$$H_p(\beta)=H_{p_0}+H_{p_1}\beta_1+\cdots+H_{p_m}\beta_m \tag{67}$$

$$=\tilde{H}_p\Lambda(\beta) \tag{68}$$

with the β_i being analogous to α_i. This would facilitate the retrofitting of existing fixed assets (unit processes) to allow for flexible manufacturing processes by varying the interconnection structure, without redesigning the control system. See Ref. 35 for examples of recent problems and developments in this area. Such an approach allows for more efficient use of plant and equipment, it also allows for the process network to continue operation even if some units are shut down for maintenance. An area of application may be in renewable energy distributed networks, whereby individual nodes may act alternately as net producers or consumers of energy.

Another possible extension of the framework presented in this article is to control of systems with varying delay in the process (due to flow rates changing with operational modes) and controller (due to network traffic) networks. Communication issues such as time delay in networked control systems have received attention in the recent literature.[36,37] In this case, the time delays can be treated as a part of the networks for analysis in a similar manner to the parameters α_i and β_i, as forward shift operators of varying degree. In which case, the process and controller networks become dynamic systems. This is facilitated using QdF supply rates in the discrete time setting as the indeterminates ζ and η in the QdFs are forward shift operators. As such, it is possible to determine the dissipativity properties of the delayed system from that of the delay-free system using polynomial matrix operations, some preliminary results on the analysis of such systems appear in Ref. 38.

To extend the approach presented in this work to set point tracking, it is necessary to develop an approach based on nonlinear processes, as the linear approximations may no longer be valid. A key hurdle in this approach is the determination of the dissipativity properties of the nonlinear processes. For input affine systems and (Q, S, R)-type supply rates, there are existing approaches in the literature, for example, Refs. 39 and 40. However, to the best of the authors knowledge, there are currently no analogous results for QdF supply rates. One possible approach, however, is to extend the parameterization presented in Ref. 41, whereby the QdF dissipativity of the process can be implied by the (Q, S, R) dissipativity of a related system.

Acknowledgments

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Appendix

Problem 3 (*Tippett and Bao*[14]). *The algorithm for the ith controller with hard input constraints and soft output constraints has local output, $\hat{y}_L(k)$, which is the minimizer of the ith controllers storage function, ϕ_i. At any time step k it is as follows:*

1. *If $k=0$ set $\hat{u}_r(k)=0$ (the input from remote controllers). If $k>0$ and this is the first iteration (of this time step) set $\hat{u}_r(k)$ as the last one received last time step shifted forward one time step, $\hat{u}_r(k)=\sigma\hat{u}_r(k-1)$. Otherwise set $\hat{u}_r(k)$ that which is received from the other controllers.*

2. *Optimize local input by the following LMI problem*

$$\hat{y}_L(k)=\text{argmin } \gamma \tag{A1}$$

subject to

$$\begin{pmatrix} \left[\hat{D}_{1_i}^T\psi_{11_i}\hat{D}_{1_i}+\hat{D}_{1_i}^T\psi_{12_i}+\psi_{12_i}^T\hat{D}_{1_i}+\psi_{22_i}\right]^{-1} & \hat{y}_L(k) \\ \hat{y}_L^T(k) & \gamma+\Lambda_i+w_y\epsilon_i \end{pmatrix} \geq 0 \tag{A2}$$

$$\begin{pmatrix} -\chi_{i_{ll}}^{-1} & \hat{y}_L(k) \\ \hat{y}_L^T(k) & \Omega_{i_{ll}}+W_{k-n-1} \end{pmatrix} \geq 0 \tag{A3}$$

$$-\chi_{i_{ll}} > 0 \tag{A4}$$

$$\hat{y}_L(k) \in \mathcal{U} \tag{A5}$$

$$\begin{pmatrix} P_i^{-1} & \hat{C}_i x_i(k)+\hat{D}_{1_i}\hat{y}_L+\hat{D}_{2_i}\hat{u}_r \\ \left(\hat{C}_i x_i(k)+\hat{D}_{1_i}\hat{y}_L+\hat{D}_{2_i}\hat{u}_r\right)^T & 1+\epsilon_i \end{pmatrix} \geq 0 \tag{A6}$$

$$\epsilon_i \geq 0 \tag{A7}$$

where $\mathcal{U}$ is any convex set containing the origin such that the constraint $\hat{y}_L(k)\in\mathcal{U}$ may be written as a LMI, the (ellipsoid) output constraints are $\hat{y}^T(k)P_i\hat{y}(k)\leq 1+\epsilon_i$ with $P_i>0$, $\epsilon_i\geq 0\ \forall i$ and

$$\begin{aligned}\Lambda_i=&-x_i^T(k)\hat{C}_i^T\psi_{11_i}\hat{C}_i x_i(k)-2x_i^T(k)\hat{C}_i^T\left(\psi_{11_i}\hat{D}_{1_i}+\psi_{12_i}\right)\hat{y}_L\\ &-2x_i^T(k)\hat{C}_i^T\left(\psi_{11_i}\hat{D}_{2_i}+\psi_{14_i}\right)\hat{u}_r\\ &-2\hat{y}_L^T\left(\hat{D}_{1_i}^T\psi_{11_i}\hat{D}_{2_i}+\psi_{24_i}+\psi_{12_i}^T\hat{D}_{2_i}^T+\hat{D}_{1_i}^T\psi_{14_i}\right)\hat{u}_r\\ &-\hat{u}_r^T\left(\hat{D}_{2_i}^T\psi_{11_i}\hat{D}_{2_i}+\hat{D}_{2_i}^T\psi_{14_i}+\psi_{14_i}^T\hat{D}_{2_i}+\psi_{44_i}\right)\hat{u}_r\end{aligned} \tag{A8}$$

$$\chi_{i_{ll}}=\begin{pmatrix} Q_{11_i}+S_{11_i}^L D_{1_i}^L+D_{1_i}^{L^T}R_{11_i}^{LL}D_{1_i}^L & Q_{10_i}+S_{11_i}^L D_{2_i}^L+D_{1_i}^{L^T}S_{01_i}^{L^T}+D_{1_i}^{L^T}R_{11_i}^{LL}D_{2_i}^L \\ \left(Q_{10_i}+S_{11_i}^L D_{2_i}^L+D_{1_i}^{L^T}S_{01_i}^{L^T}+D_{1_i}^{L^T}R_{11_i}^{LL}D_{2_i}^L\right)^T & Q_{00_i}+Q_{11_i}+S_{01_i}^L D_{2_i}^L+D_{2_i}^{L^T}R_{11_i}^{LL}D_{2_i}^L \end{pmatrix} \tag{A9}$$

3. *Calculate and send to remote controller(s) predicted local process output*

$$\hat{y}_i = \hat{C}x_i(k) + \hat{D}_{1_i}\hat{u}_{L_i} + \hat{D}_{2_i}\hat{u}_{p_i} \tag{A10}$$

4. *If number of iterations is greater or equal to $I_{\max}$ cease iteration and apply optimal output in Step 2. Otherwise, return to Step 1.*

Ignoring external disturbances, the storage function of the ith process is given by

$$Q_{\psi_i}(k) = \begin{pmatrix} \hat{y}(k) \\ \hat{y}_L(k) \\ \hat{u}_r(k) \end{pmatrix}^T \begin{pmatrix} \tilde{\psi}_{11} & \tilde{\psi}_{12} & \tilde{\psi}_{14} \\ \tilde{\psi}_{12}^T & \tilde{\psi}_{22} & \tilde{\psi}_{24} \\ \tilde{\psi}_{14^T} & \tilde{\psi}_{24^T} & \tilde{\psi}_{44} \end{pmatrix} \begin{pmatrix} \hat{y}(k) \\ \hat{y}_L(k) \\ \hat{u}_r(k) \end{pmatrix} \tag{A11}$$

To take any soft output constraints into account the cost function is chosen as $J(k) = Q_{\psi_i}(k) + w_y\epsilon$. By choosing a $\gamma \geq 0$ such that $\gamma - J(k) \geq 0$, the minimization of the cost function may be transformed into the LMI constraint (A2). Meanwhile Constraints (A5–A7) ensure the satisfaction of the hard (convex) input and soft input constraints. Finally Constraints (A3) and (A4) ensure that the ith controller traces a dissipative trajectory, the details of this constraint are described below

Formulation of dissipative trajectory constraint

Permute the coefficient matrix of $\phi_c(\zeta, \eta)$ (the matrix which induces the supply rate of the controller) conformally with the controller inputs and outputs to be $P^T\tilde{\phi}_c P = \begin{pmatrix} \tilde{Q}_c & \tilde{S}_c \\ \tilde{S}_c^T & \tilde{R}_c \end{pmatrix}$. For the current time k, the controller traces a dissipative trajectory if

$$\sum_{i=0}^{k} Q_{\phi_c} \geq 0 \tag{A12}$$

this condition becomes

$$W_{k-N-1} + Q_{\phi_{c_l}}(k-N) + Q_{\phi_{c_l}}(k-N+1) + \cdots + Q_{\phi_{c_m}}(k) \geq 0 \tag{A13}$$

where W_{k-N-1} is a constant, the sum of all previous values of the controller supply rate (with $W_{k-N-1} \geq 0$, this is ensured if $W_k = 0$ and the given condition is satisfied for all $k > 0$) and the other terms relate to the current and recent values of the controller supply rate, which are decision variables. Using $\hat{u}_R = \hat{y}_L$, $Q_{\tilde{\phi}_c}(k)$ becomes

$$y_L^T(k)Q_c^{ll}y_L(k) + 2y_L^T(k)\begin{pmatrix} S_c^{ll} + \hat{Q}_c^{lr} & S_{c_m}^{lr} \end{pmatrix}\begin{pmatrix} u_L(k) \\ u_R(k) \end{pmatrix}$$
$$+ \begin{pmatrix} u_L(k) \\ u_R(k) \end{pmatrix}^T \begin{pmatrix} R_c^{ll} + Q_c^{rr} + S_c^{rl} + S_c^{rl^T} & R_c^{lr} + S_c^{rr} \\ R_c^{lr^T} + S_c^{rr^T} & \hat{R}_c^{rr} \end{pmatrix} \begin{pmatrix} u_L(k) \\ u_R(k) \end{pmatrix} \geq 0 \tag{A14}$$

Let $Q_c^{ll} = Q$, $\begin{pmatrix} S_c^{ll} + Q_c^{lr} & S_c^{lr} \end{pmatrix} = \begin{pmatrix} S^l & S^r \end{pmatrix}$ and

$$\begin{pmatrix} R_c^{ll} + Q_c^{rr} + S_c^{rl} + S_c^{rl^T} & R_c^{lr} + S_c^{rr} \\ R_c^{lr^T} + S_c^{rr^T} & \hat{R}_c^{rr} \end{pmatrix} = \begin{pmatrix} R^{ll} & R^{lr} \\ R^{lr^T} & R^{rr} \end{pmatrix} \tag{A15}$$

The controller dissipativity condition, (A13), may then be written as

$$W_{k-N-1} + \begin{pmatrix} \hat{y}_L(k) \\ \vdots \\ \hat{y}_L(k-n) \\ \hat{u}_L(k) \\ \vdots \\ \hat{u}_L(k-n) \\ \hat{u}_R(k) \\ \vdots \\ \hat{u}_R(k-N) \end{pmatrix}^T \begin{pmatrix} Q & S^l & S^r \\ S^{l^T} & R^{ll} & R^{lr} \\ S^{r^T} & R^{lr^T} & R^{rr} \end{pmatrix} \begin{pmatrix} \hat{y}_L(k) \\ \vdots \\ \hat{y}_L(k-N) \\ \hat{u}_L(k) \\ \vdots \\ \hat{u}_L(k-N) \\ \hat{u}_R(k) \\ \vdots \\ \hat{u}_R(k-N) \end{pmatrix} \geq 0 \tag{A16}$$

As the $\tilde{u}_L(k+1)$ is predicted using the model of the (local) process

$$\tilde{u}_L(k+1) = \hat{C}x(k) + D_1\tilde{y}_L(k+1) + CB_1y_L(k) + D_3\tilde{u}_R'(k+1) + CB_3\tilde{u}_R'(k) \tag{A17}$$

These vectors "overlap" in the sense that (assuming a supply rate of at least order one) $\hat{y}_L(k-1)$ and $\hat{y}_L(k)$ are both functions of $y_L(k)$, with the same relationship holding for $\hat{u}_L$ and $\hat{u}_R$. Once these future steps are realized, the predicted values are equated with the actual values. For ease of exposition, this is shown below in the original, not extended, variable space for the case of a first-order supply rate and the case where the controller switched supply rates the previous time step (although it is analogous for higher orders).

$$W_{k-2} + \begin{pmatrix} y_L(k+1) \\ y_L(k) \\ y_L(k-1) \\ u_L(k+1) \\ u_L(k) \\ u_L(k-1) \\ u_R(k+1) \\ u_R(k) \\ u_R(k-1) \end{pmatrix}^T \begin{pmatrix} Q_{11} & Q_{10} & 0 & S_{11}^l & S_{10}^l & 0 & S_{11}^r & S_{10}^r & 0 \\ * & Q_{00}+Q_{11} & Q_{10} & S_{01}^l & S_{00}^l + S_{11}^l & S_{10}^l & S_{01}^r & S_{00}^r + S_{11}^r & S_{10}^r \\ * & * & Q_{00} & 0 & S_{01}^l & S_{00}^l & 0 & S_{01}^r & S_{00}^r \\ * & * & * & R_{11}^{ll} & R_{10}^{ll} & 0 & R_{11}^{lr} & R_{10}^{ll} & 0 \\ * & * & * & * & R_{00}^{ll} + R_{11}^{ll} & R_{10}^{ll} & R_{01}^{lr} & R_{00}^{lr} + R_{11}^{lr} & R_{10}^{lr} \\ * & * & * & * & * & R_{00}^{ll} & 0 & R_{01}^{lr} & R_{00}^{lr} \\ * & * & * & * & * & * & R_{11}^{rr} & R_{10}^{rr} & 0 \\ * & * & * & * & * & * & * & R_{00}^{rr} + R_{11}^{rr} & R_{10}^{rr} \\ * & * & * & * & * & * & * & * & R_{00}^{rr} \end{pmatrix} \begin{pmatrix} y_L(k+1) \\ y_L(k) \\ y_L(k-1) \\ u_L(k+1) \\ u_L(k) \\ u_L(k-1) \\ u_R(k+1) \\ u_R(k) \\ u_R(k-1) \end{pmatrix} \geq 0 \tag{A18}$$

where vectors at time $k+1$ are predicted values of the local control output, local process output, and input from remote controllers. $u_L(k+1)$, the controllers prediction of the output of its local process at time $k+1$, may be eliminated using the model of the (local) process (A17) where the $u_R^{'}$ denotes the prediction of remote process outputs from the last iteration of communication

$$W_{k-2}+\begin{pmatrix} \hat{C}x(k) \\ y_L(k+1) \\ y_L(k) \\ y_L(k-1) \\ u_L(k) \\ u_L(k-1) \\ u_R(k+1) \\ u_R(k) \\ u_R(k-1) \\ u_R^{'}(k+1) \\ u_R^{'}(k) \end{pmatrix}^T \begin{pmatrix} Z_{11} & Z_{12} & Z_{13} & Z_{14} & Z_{15} & Z_{16} & Z_{17} & Z_{18} & Z_{19} & Z_{110} & Z_{111} \\ * & Z_{22} & Z_{23} & Z_{24} & Z_{25} & Z_{26} & Z_{27} & Z_{28} & Z_{29} & Z_{210} & Z_{211} \\ * & * & Z_{33} & Z_{34} & Z_{35} & Z_{36} & Z_{37} & Z_{38} & Z_{39} & Z_{310} & Z_{311} \\ * & * & * & Z_{44} & Z_{45} & Z_{46} & Z_{47} & Z_{48} & Z_{49} & Z_{410} & Z_{411} \\ * & * & * & * & Z_{55} & Z_{56} & Z_{57} & Z_{58} & Z_{59} & Z_{510} & Z_{511} \\ * & * & * & * & * & Z_{66} & Z_{67} & Z_{68} & Z_{69} & Z_{610} & Z_{611} \\ * & * & * & * & * & * & Z_{77} & Z_{78} & Z_{79} & Z_{710} & Z_{711} \\ * & * & * & * & * & * & * & Z_{88} & Z_{89} & Z_{810} & Z_{811} \\ * & * & * & * & * & * & * & * & Z_{99} & Z_{910} & Z_{911} \\ * & * & * & * & * & * & * & * & * & Z_{1010} & Z_{1011} \\ * & * & * & * & * & * & * & * & * & * & Z_{1111} \end{pmatrix} \begin{pmatrix} \hat{C}x(k) \\ y_L(k+1) \\ y_L(k) \\ y_L(k-1) \\ u_L(k) \\ u_L(k-1) \\ u_R(k+1) \\ u_R(k) \\ u_R(k-1) \\ u_R^{'}(k+1) \\ u_R^{'}(k) \end{pmatrix} \geq 0 \tag{A19}$$

Or

$$W_{k-2}+\Omega \geq 0 \tag{A20}$$

Alternatively this may be written as

$$W_{k-2}+\begin{pmatrix} \hat{C}x(k) \\ y_L(k+1) \\ y_L(k) \\ y_L(k-1) \\ u_L(k) \\ u_L(k-1) \\ u_R(k+1) \\ u_R(k) \\ u_R(k-1) \\ u_R^{'}(k+1) \\ u_R^{'}(k) \end{pmatrix}^T \begin{pmatrix} Z_{11} & Z_{12} & Z_{13} & Z_{14} & Z_{15} & Z_{16} & Z_{17} & Z_{18} & Z_{19} & Z_{110} & Z_{111} \\ * & 0 & 0 & Z_{24} & Z_{25} & Z_{26} & Z_{27} & Z_{28} & Z_{29} & Z_{210} & Z_{211} \\ * & * & 0 & Z_{34} & Z_{35} & Z_{36} & Z_{37} & Z_{38} & Z_{39} & Z_{310} & Z_{311} \\ * & * & * & Z_{44} & Z_{45} & Z_{46} & Z_{47} & Z_{48} & Z_{49} & Z_{410} & Z_{411} \\ * & * & * & * & Z_{55} & Z_{56} & Z_{57} & Z_{58} & Z_{59} & Z_{510} & Z_{511} \\ * & * & * & * & * & Z_{66} & Z_{67} & Z_{68} & Z_{69} & Z_{610} & Z_{611} \\ * & * & * & * & * & * & Z_{77} & Z_{78} & Z_{79} & Z_{710} & Z_{711} \\ * & * & * & * & * & * & * & Z_{88} & Z_{89} & Z_{810} & Z_{811} \\ * & * & * & * & * & * & * & * & Z_{99} & Z_{910} & Z_{911} \\ * & * & * & * & * & * & * & * & * & Z_{1010} & Z_{1011} \\ * & * & * & * & * & * & * & * & * & * & Z_{1111} \end{pmatrix} \begin{pmatrix} \hat{C}x(k) \\ y_L(k+1) \\ y_L(k) \\ y_L(k-1) \\ u_L(k) \\ u_L(k-1) \\ u_R(k+1) \\ u_R(k) \\ u_R(k-1) \\ u_R^{'}(k+1) \\ u_R^{'}(k) \end{pmatrix} + \begin{pmatrix} \tilde{y}_L(k+1) \\ y_L(k) \end{pmatrix}^T \begin{pmatrix} Z_{22} & Z_{23} \\ Z_{23}^T & Z_{33} \end{pmatrix} \begin{pmatrix} \tilde{y}_L(k+1) \\ y_L(k) \end{pmatrix} \geq 0 \tag{A21}$$

where the differences have been bolded, this may be written as

$$W_{k-2}+\Omega+\begin{pmatrix} \tilde{y}_L(k+1) \\ y_L(k) \end{pmatrix}^T \begin{pmatrix} Z_{22} & Z_{23} \\ Z_{23}^T & Z_{33} \end{pmatrix} \begin{pmatrix} \tilde{y}_L(k+1) \\ y_L(k) \end{pmatrix} \geq 0 \tag{A22}$$

with

$Z_{11}=R_{11}^{LL}$ $\quad Z_{12}=S_{11}^{L^T}+R_{11}^{LL}D_1$

$Z_{13}=S_{01}^{L^T}+R_{11}^{LL}CB_1$ $\quad Z_{14}=0$

$Z_{15}=R_{10}^{LL}$ $\quad Z_{16}=0$

$Z_{17}=R_{11}^{LR}$ $\quad Z_{18}=R_{10}^{LR}$

$Z_{19}=0$ $\quad Z_{110}=R_{11}^{LL}D_3$

$Z_{111}=R_{11}^{LL}CB_3$ $\quad Z_{22}=Q_{11}+S_{11}^{L}D_1+D_1^TR_{11}^{LL}D_1$

$Z_{23}=Q_{10}+S_{11}^{L}CB_1+D_1^TS_{01}^{L^T}+D_1^TR_{11}^{LL}CB_1$ $\quad Z_{24}=0$

$Z_{25}=S_{10}^{L}+D_1^TR_{10}^{LL}$ $\quad Z_{26}=0$

$Z_{27}=S_{11}^{R}+D_1^TR_{11}^{LR}$ $\quad Z_{28}=S_{10}^{R}+D_1^TR_{10}^{LR}$

$Z_{29}=0$ $\quad Z_{210}=S_{11}^{L}D_3+D_1^TR_{11}^{LL}D_3$

$Z_{211}=S_{11}^{L}CB_3+D_1^TR_{11}^{L}CB_3$ $\quad Z_{33}=Q_{00}+Q_{11}+S_{01}^{L}CB_1+(CB_1)^TR_{11}^{LL}CB_1$

$Z_{34}=Q_{10_l}$ $\quad Z_{35}=S_{00}^{L}+S_{11}^{L}+(CB_1)^TR_{10}^{LL}$

$Z_{36}=S_{10}^{L}$ $\quad Z_{37}=S_{01}^{R}+(CB_1)^TR_{11}^{LR}$

$Z_{38}=S_{00}^{R}+S_{11}^{R}+(CB_1)^TR_{10}^{LR}$ $\quad Z_{39}=S_{10}^{R}$

$Z_{310}=S_{10}^{L}D_3+(CB_1)^TR_{11}^{LL}D_3$ $\quad Z_{311}=S_{01}^{L}CB_3+(CB_1)^TR_{11}^{LL}D_3$

$Z_{44}=Q_{00}$ $\quad Z_{45}=S_{01}^{L}$

$Z_{46}=S_{00}^{L}$ $\quad Z_{47}=0$

$Z_{48}=S_{01}^{R}$ $\quad Z_{49}=S_{00}^{R}$

$Z_{410}=0$ $\quad Z_{411}=0$

$Z_{55}=R_{00}^{LL}+R_{11}^{LL}$ $\quad Z_{56}=R_{10}^{LL}$

$Z_{57}=R_{01}^{LR}$ $\quad Z_{58}=R_{00}^{LR}+R_{11}^{LR}$

$Z_{59}=R_{10}^{LR}$ $\quad Z_{510}=R_{10}^{LL^T}D_3$

$Z_{511}=R_{10}^{LL^T}CB_3$ $\quad Z_{66}=R_{00}^{LL}$

$Z_{67}=0$ $\quad Z_{68}=R_{01}^{LR}$

$Z_{69}=R_{00}^{LR}$ $\quad Z_{610}=0$

$Z_{611}=0$ $\quad Z_{77}=R_{11}^{RR}$

$Z_{78}=R_{10}^{RR}$ $\quad Z_{79}=0$

$Z_{710}=R_{11}^{LR^T}D_3$ $\quad Z_{711}=R_{11}^{LR^T}CB_3$

$Z_{88}=R_{00}^{RR}+R_{11}^{RR}$ $\quad Z_{89}=R_{10}^{RR}$

$Z_{810}=R_{10}^{LR^T}D_3$ $\quad Z_{811}=R_{10}^{LR^T}CB_3$

$Z_{99}=R_{00}^{RR}$ $\quad Z_{910}=0$

$Z_{911}=0$ $\quad Z_{1010}=D_3^TR_{11}^{LL}D_3$

$Z_{1011}=D_3^TR_{11}^{LL}CB_3$ $\quad Z_{1111}=(CB_3)^TR_{11}^{LL}CB_3$

Letting

$$\chi=\begin{pmatrix} Z_{22} & Z_{23} \\ Z_{23}^T & Z_{33} \end{pmatrix} = \begin{pmatrix} Q_{11}+S_{11}^{L}D_1+D_1^TR_{11}^{LL}D_1 & Q_{10}+S_{11}^{L}CB_1+D_1^TS_{01}^{L^T}+D_1^TR_{11}^{LL}CB_1 \\ \left(Q_{10}+S_{11}^{L}CB_1+D_1^TS_{01}^{L^T}+D_1^TR_{11}^{LL}CB_1\right)^T & Q_{00}+Q_{11}+S_{01}^{L}CB_1+(CB_1)^TR_{11}^{LL}CB_1 \end{pmatrix} < 0 \tag{A23}$$

and taking a Schur complement in that block in (A22), we arrive at the stated LMI constraint. Thus, this constraint implies the controller traces a dissipative trajectory when the controller supply rate switches.

Proposition 5 (*Tippett and Bao*[14]). *A sufficient condition for the dissipative distributed MPC algorithm in Problem 3 to be feasible at any time* $k>0$ *for either the unconstrained case or with hard input constraints is that the trajectory condition is originally feasible at* $t=0$ ($W_0 \geq 0$)*, the constraint set* $u_c(k) \in \mathcal{U}$ *contains the origin,* $\Omega_i \geq 0$*, and that*

$$\hat{D}_{1_i}^T\psi_{11_i}\hat{D}_{1_i}+\hat{D}_{1_i}^T\psi_{12_i}+\psi_{12_i}^T\hat{D}_{1_i}+\psi_{22_i} > 0 \tag{A24}$$

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